

Centre de Physique Théorique\* - CNRS - Luminy, Case 907  
F-13288 Marseille Cedex 9 - France

## Killing Form on Quasitriangular Hopf Algebras and Quantum Lie Algebras

Paul WATTS<sup>†</sup>

### Abstract

The basics of quasitriangular Hopf algebras and quantum Lie algebras are briefly reviewed, and it is shown that their properties allow the introduction of a Killing form. For quantum Lie algebras, this leads to the definitions of a Killing metric and quadratic casimir. The specific case of  $U_q(\mathfrak{su}(N))$  is examined in detail, where it is shown that many of the classical results are reproduced, and explicit calculations to illustrate the conclusions are presented for  $U_q(\mathfrak{su}(2))$ .

MSC-91: 16W30, 17B37, 81R50

PACS-95: 02.20.Sv

May 1995

CPT-95/P.3201

Anonymous FTP or gopher: *cpt.univ-mrs.fr*

---

\*Unité Propre de Recherche 7061

<sup>†</sup>E-mail: *watts@cpt.univ-mrs.fr* - WWW Home Page: *http://cpt.univ-mrs.fr/~watts*

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Quasitriangular Hopf Algebras</b>	<b>2</b>
2.1	Representations of Hopf Algebras . . . . .	4
2.2	Representations of Quasitriangular Hopf Algebras and Quantum Groups . . . . .	5
<b>3</b>	<b>Actions and Coactions</b>	<b>7</b>
3.1	Actions . . . . .	7
3.2	Coactions . . . . .	8
<b>4</b>	<b>Quantum Lie Algebras</b>	<b>9</b>
4.1	Basics of Quantum Lie Algebras . . . . .	9
4.2	The Adjoint Representation . . . . .	11
4.3	Quasitriangular Quantum Lie Algebras . . . . .	12
<b>5</b>	<b>The Killing Form</b>	<b>14</b>
5.1	The Killing Form for a Quasitriangular Hopf Algebra . . . . .	14
5.2	The Killing Metric for a Quantum Lie Algebra . . . . .	16
5.3	Quadratic Casimirs . . . . .	17
5.4	The Canonical Killing Metric . . . . .	18
5.5	Comments on the Adjoint Representation . . . . .	19
<b>6</b>	<b>The Quantum Lie Algebra <math>U_q(\mathfrak{su}(N))</math></b>	<b>21</b>
6.1	R-Matrix Construction . . . . .	21
6.2	Properties of the Killing Metric . . . . .	24
6.3	Hopf Algebraic Structure . . . . .	26
6.4	Other Quantum Lie Algebras . . . . .	27
<b>7</b>	<b>Example: <math>U_q(\mathfrak{su}(2))</math></b>	<b>28</b>
7.1	Fundamental Representation . . . . .	30
7.2	Adjoint Representation . . . . .	32
<b>8</b>	<b>Conclusions</b>	<b>33</b>
<b>A</b>	<b>Appendix: The Matrix <math>D</math></b>	<b>36</b>



# 1 Introduction

In studying an algebra  $\mathfrak{g}$ , one often finds it useful to consider the Killing metric on  $\mathfrak{g}$ . From a mathematical perspective, this allows some insight into some of the algebra's properties. For instance,  $\mathfrak{g}$  is a compact Lie group if the Killing metric associated with the adjoint representation is positive definite. As another example, it is often the case that the Killing metric computed from the trace over a given representation ("rep") may say something about its reducibility or irreducibility. It also figures prominently in the analysis of certain infinite-dimensional Lie algebras; if one has a finite-dimensional Lie algebra  $\mathfrak{g}$ , then the central extension which appears in the commutation relations of the affine algebra  $\widehat{\mathfrak{g}}$  depends explicitly upon the Killing metric of  $\mathfrak{g}$ .

However, from a physicist's point of view, the importance of this object does not lie so much in its usefulness as a way of studying the abstract structure of an algebra, but rather as a tool in constructing a physical theory. Perhaps the most obvious example of this is when one wants to find quadratic casimirs; recall that the quadratic casimir of a Lie algebra  $\mathfrak{g}$  is central within the universal enveloping algebra  $U(\mathfrak{g})$ . Therefore, given an irreducible representation ("irrep") of  $\mathfrak{g}$ , the matrix corresponding to the quadratic casimir is proportional to the identity, and the constant of proportionality labels the irrep. In constructing a physical theory for which  $\mathfrak{g}$  is a symmetry algebra, one (generally) requires that a particle in the spectrum of our theory lives in an irrep of  $\mathfrak{g}$ , and therefore the good quantum numbers describing this particle will include not only the weights of the irrep, but also the value of the quadratic casimir. (The most familiar example of this is  $\mathfrak{g} = \mathfrak{su}(2)$ , where the quadratic casimir is  $\vec{J}^2$ , which of course takes the value  $j(j+1)$  in the spin  $j$  irrep.) These casimirs are constructed (for the cases where  $\mathfrak{g}$  is semisimple) using the Killing metric, and therefore the utility of finding such a metric is obvious.

Another very important way in which the Killing metric appears is in the construction of lagrangeans, in the following fashion: suppose one wishes to construct a Yang-Mills theory in which  $\mathfrak{g}$  is the Lie algebra of the gauge group. If  $\Gamma$  is the gauge field 1-form, and  $\rho$  the rep in which it lives, the

action is (up to a multiplicative constant)

$$S = \int \text{tr}_\rho(F \wedge \star F) + \dots, \quad (1.1)$$

where  $F = d\Gamma + \Gamma \wedge \Gamma$  is the field strength 2-form. If  $\Gamma \equiv \rho(T_A)\Gamma^A$ , where  $\{T_A\}$  are the generators of  $\mathfrak{g}$ , then one sees the appearance of the Killing metric  $\eta_{AB}^{(\rho)} := \text{tr}_\rho(T_A T_B)$  associated with this rep:

$$S = \int \eta_{AB}^{(\rho)} F^A \wedge \star F^B + \dots \quad (1.2)$$

Thus, knowledge of the Killing metric is vital to the construction of a Yang-Mills theory, since it shows up explicitly in the gauge field kinetic energy term.

The purpose of this work is to extend the concept of a Killing metric to the case where the algebra in question is a quasitriangular Hopf algebra. As just stated, such a metric is of extreme utility when one considers a physical theory which has some global or local symmetry, so if one wants to formulate such a theory where the symmetry algebra is a *quantum* rather than a classical one, and examine its particle content, scattering amplitudes, *etc.*, the Killing metric will be of great importance.

The sections immediately following this Introduction will serve principally to establish the language and notation which appear throughout this paper, namely, those of Hopf algebras and quantum Lie algebras, and should be treated as brief reviews, since the majority of the material therein may be found in more detail elsewhere. The main thrust of this work starts in Section 5, where the Killing form on an arbitrary quasitriangular Hopf algebra is first introduced, and then the Killing metric is found by specifying to the case of a quantum Lie algebra.  $U_q(\mathfrak{su}(N))$ , and specifically  $U_q(\mathfrak{su}(2))$ , will then be treated in order to illustrate the connections with the classical case and provide concrete examples of the results. In addition, there is an Appendix containing useful information about the element  $u$  which plays such an important role in the formulation of the Killing form.

## 2 Quasitriangular Hopf Algebras

Not surprisingly, given the subject of this paper, it will be necessary to review the basics of the representation theory of Hopf algebras and quasitriangular

Hopf algebras (the latter of which gives rise to quantum groups). First, a very brief reminder: a *Hopf algebra (HA)*  $\mathfrak{U}$  is a unital associative algebra (with unit 1) over a field  $k^1$  on which there exist the following linear maps: the *coproduct* (or *comultiplication*)  $\Delta : \mathfrak{U} \rightarrow \mathfrak{U} \otimes \mathfrak{U}$ , the *counit*  $\epsilon : \mathfrak{U} \rightarrow k$ , and the *antipode*  $S : \mathfrak{U} \rightarrow \mathfrak{U}$ , which satisfy the conditions

$$\begin{aligned} (\Delta \otimes \text{id})\Delta(x) &= (\text{id} \otimes \Delta)\Delta(x), & \Delta(xy) &= \Delta(x)\Delta(y), \\ (\epsilon \otimes \text{id})\Delta(x) &= (\text{id} \otimes \epsilon)\Delta(x) = x, & \epsilon(xy) &= \epsilon(x)\epsilon(y), \\ \cdot(S \otimes \text{id})\Delta(x) &= \cdot(\text{id} \otimes S)\Delta(x) = 1\epsilon(x), \end{aligned} \quad (2.1)$$

$x, y \in \mathfrak{U}$ . Furthermore, if a  $*$ -structure is desired, the involution  $\theta : \mathfrak{U} \rightarrow \mathfrak{U}$  must be an idempotent antilinear map such that

$$\theta(S(\theta(x))) = S^{-1}(x), \quad \epsilon(\theta(x)) = \epsilon(x)^*, \quad \Delta(\theta(x)) = (\theta \otimes \theta)(\Delta(x)), \quad (2.2)$$

where  $*$  is the conjugation in  $k$ . (These are just the defining properties; if the reader is interested in learning more about HAs, s/he may refer to [1, 2, 3].)

Before proceeding to the next subsection, it is necessary to introduce some terminology: by the “rep of a HA  $\mathfrak{U}$ ”, I will always mean a 1-to-1 linear map  $\rho$  from  $\mathfrak{U}$  to some finite-dimensional matrix group over the field  $k$  such that  $\rho(xy) = \rho(x)\rho(y)$ , just as in the classical case. A “irrep” is a nonreducible rep of  $\mathfrak{U}$ , with reducibility also being defined in the same sense as in the classical case, *i.e.* the existence of a common nonzero null eigenvector for all matrices in the rep. The reduction of a rep  $\rho$  to irreps  $\{\rho_i\}$  as a direct sum  $\bigoplus_i \rho_i$  then follows accordingly.

Notice that nowhere in these definitions has there been any mention of anything but the structure of  $\mathfrak{U}$  as an *algebra*; the fact that  $\mathfrak{U}$  is a *Hopf algebra* appears when one considers tensor product reps of  $\mathfrak{U}$ , whose constructions are defined via the coproduct, namely, if  $\rho^i_j$  and  $\hat{\rho}^A_B$  are two reps of  $\mathfrak{U}$ , then the tensor product rep  $\rho \otimes \hat{\rho}$  is defined by

$$(\rho \otimes \hat{\rho})^{iA}_{jB}(x) := \rho^i_j(x_{(1)})\hat{\rho}^A_B(x_{(2)}), \quad (2.3)$$

where  $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$  is the very useful Sweedler notation for the coproduct [1]. Therefore, the HA structure will emerge when, for example, one wants to find the irreps  $\{\rho_i\}$  which appear in the decomposition  $\rho \otimes \hat{\rho} = \bigoplus_i \rho_i$ .

---

<sup>1</sup>Throughout this work, the term “numerical” is equivalent to “ $k$ -valued”.

One final note for this subsection: when the deformation parameter  $q$  is introduced later, it will always be treated as a formal variable; in other words, as an arbitrary element of the field  $k$ . There exist situations, however, where the actual value of  $q$  is important, *e.g.* when  $k = \mathbb{C}$ , the quantity  $1 + q^2$  vanishes when  $q = \pm i$ , and this may cause certain expressions to blow up, or irreps to suddenly become reducible, or the like. In this work, however, these situations will not be considered.

## 2.1 Representations of Hopf Algebras

Two HAs (both over the same field  $k$ )  $\mathfrak{U}$  and  $\mathfrak{A}$  are said to be *dually paired* if there exists a nondegenerate inner product  $\langle \cdot, \cdot \rangle : \mathfrak{U} \otimes \mathfrak{A} \rightarrow k$  such that their respective units, coproducts, counits, and antipodes satisfy

$$\begin{aligned} \langle xy, a \rangle &= \langle x \otimes y, \Delta(a) \rangle, \\ \langle 1, a \rangle &= \epsilon(a), \\ \langle \Delta(x), a \otimes b \rangle &= \langle x, ab \rangle, \\ \epsilon(x) &= \langle x, 1 \rangle, \\ \langle S(x), a \rangle &= \langle x, S(a) \rangle, \end{aligned} \tag{2.4}$$

and if a  $*$ -structure exists, the involutions satisfy

$$\langle \theta(x), a \rangle = \langle x, \theta(S(a)) \rangle^*, \tag{2.5}$$

where  $x, y \in \mathfrak{U}$  and  $a, b \in \mathfrak{A}$ . (It easy to show that all the relevant consistency relations to ensure that the two algebras are indeed HAs are satisfied.)

One may also use this dual pairing constructively: if given a HA  $\mathfrak{U}$ , together with a  $N \times N$  matrix rep  $\rho : \mathfrak{U} \rightarrow M_N(k)$ , it is possible to define another HA dually paired with  $\mathfrak{U}$  [4]; this new HA  $\mathfrak{A}$  is taken to be the one generated by the  $N^2$  elements  $A^i_j$  defined through

$$\rho^i_j(x) \equiv \langle x, A^i_j \rangle. \tag{2.6}$$

The faithfulness of the rep guarantees that this inner product is nondegenerate, and thus that the elements of the matrix  $A$  are uniquely determined; furthermore, the fact that  $\rho$  is an algebra map immediately gives

$$\Delta(A^i_j) = A^i_k \otimes A^k_j, \epsilon(A^i_j) = \delta^i_j, S(A^i_j) = (A^{-1})^i_j. \tag{2.7}$$

The multiplication on  $\mathfrak{A}$ , *i.e.* products between the entries of  $A$ , will of course depend upon the explicit form of the coproduct on  $\mathfrak{U}$ , through the third of (2.4).

## 2.2 Representations of Quasitriangular Hopf Algebras and Quantum Groups

A *quasitriangular Hopf algebra (QHA)* [5] is a HA  $\mathfrak{U}$  together with an invertible element, the *universal R-matrix*,  $\mathcal{R} = r_\alpha \otimes r^\alpha \in \mathfrak{U} \otimes \mathfrak{U}$  (summation implied), which satisfies the relations

$$\begin{aligned} (\Delta \otimes \text{id})(\mathcal{R}) &= \mathcal{R}_{13} \mathcal{R}_{23}, \\ (\text{id} \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{12} \mathcal{R}_{23}, \\ (\sigma \circ \Delta)(x) &= \mathcal{R} \Delta(x) \mathcal{R}^{-1}, \end{aligned} \tag{2.8}$$

where  $\sigma(x \otimes y) = y \otimes x$ , and the subscripts on  $\mathcal{R}$  are a shorthand for

$$\begin{aligned} \mathcal{R}_{12} &= r_\alpha \otimes r^\alpha \otimes 1, \\ \mathcal{R}_{13} &= r_\alpha \otimes 1 \otimes r^\alpha, \\ \mathcal{R}_{23} &= 1 \otimes r_\alpha \otimes r^\alpha. \end{aligned} \tag{2.9}$$

As a consequence of (2.8),  $\mathcal{R}$  satisfies the *quantum Yang-Baxter equation (QYBE)*

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \tag{2.10}$$

Of vital importance to the eventual introduction of the Killing form on a QHA  $\mathfrak{U}$  will be the element  $u$ , defined by

$$u := \cdot (S \otimes \text{id})(\mathcal{R}_{21}) = S(r^\alpha) r_\alpha. \tag{2.11}$$

$u$  is invertible, with inverse  $u^{-1} = r^\alpha S^2(r_\alpha)$ , and generates the square of the antipode on  $\mathfrak{U}$  by conjugation, *i.e.*

$$S^2(x) = u x u^{-1}. \tag{2.12}$$

A consequence of this property is that the element  $c := u S(u)$  is central in  $\mathfrak{U}$ .



Given a rep  $\rho$ , the commutation relations between the entries of  $A$  may be explicitly determined by using the last of the properties of the universal R-matrix from (2.8): if

$$R^{ij}_{k\ell} := \langle \mathcal{R}, A^i_k \otimes A^j_\ell \rangle \quad (2.13)$$

is the  $N^2 \times N^2$  dimensional *numerical R-matrix of  $\mathfrak{A}$* , one finds the “RAA equation” [6]

$$RA_1A_2 = A_2A_1R, \quad (2.14)$$

where the matrix indices have been replaced by the subscripts in an obvious way. It is immediate that the QYBE has a numerical counterpart, usually simply referred to as the *Yang-Baxter equation (YBE)*:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (2.15)$$

(which takes the form  $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$  when one uses the permutation matrix  $P^{ij}_{k\ell} := \delta^i_\ell \delta^j_k$  to define  $\hat{R}^{ij}_{k\ell} := (PR)^{ij}_{k\ell} \equiv R^{ji}_{k\ell}$ ).

This leads to the following definition: a HA  $\mathfrak{A}$  which is dually paired with a QHA  $\mathfrak{U}$  by means of a representation  $\rho$  in this manner is called a *matrix pseudogroup* or, more commonly nowadays, a *quantum group (QG)* [5].

(2.15) was obtained from (2.10) by applying the given rep to all three spaces of  $\mathfrak{U} \otimes \mathfrak{U} \otimes \mathfrak{U}$ , *e.g.*

$$(\rho \otimes \rho \otimes \rho)(\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}) = R_{12}R_{13}R_{23}. \quad (2.16)$$

It is also useful to consider the cases where one takes the rep in only one or two of the tensor product spaces. To see this, define the matrices  $L^\pm \in M_N(\mathfrak{U})$  by

$$\begin{aligned} L^+ &:= (\text{id} \otimes \rho)(\mathcal{R}) \equiv r_\alpha \rho(r^\alpha), \\ L^- &:= (\rho \otimes \text{id})(\mathcal{R}^{-1}) \equiv \rho(S(r_\alpha))r^\alpha. \end{aligned} \quad (2.17)$$

From the properties of  $\mathcal{R}$ , one then finds that

$$\begin{aligned} \Delta(L^\pm) &= L^\pm \dot{\otimes} L^\pm, \quad \epsilon(L^\pm) = I, \\ S(L^+) &= (L^+)^{-1} = (\text{id} \otimes \rho)(\mathcal{R}^{-1}), \quad S(L^-) = (L^-)^{-1} = (\rho \otimes \text{id})(\mathcal{R}). \end{aligned} \quad (2.18)$$

$((M \dot{\otimes} N)^i_j := M^i_k \otimes N^k_j)$ . Now, one can apply the rep to any two of the three spaces involved in (2.10) to find

$$L_1^\pm L_2^\pm R = RL_2^\pm L_1^\pm, \quad L_2^- L_1^+ R = RL_2^+ L_1^-. \quad (2.19)$$

The numerical matrices for  $L^\pm$  also follow:

$$\begin{aligned} \rho^i_j((L^+)^k_\ell) &= R^{ik}_{j\ell}, \\ \rho^i_j((L^-)^k_\ell) &= (R_{21}^{-1})^{ik}_{j\ell}. \end{aligned} \quad (2.20)$$

(As the reader shall see, these will figure very prominently in the construction of QLAs.)

### 3 Actions and Coactions

In the classical case, a Killing form for an algebra  $\mathfrak{g}$  is defined so that it is invariant under some action of  $\mathfrak{g}$  on itself (*e.g.* the adjoint action in the case of a Lie algebra); therefore, since the goal is to extend the definition to include QHAs, one must discuss the ways in which a QHA can act on itself before proceeding further in this direction. This requires introducing the concepts of actions and coactions. (A fuller discussion of the ideas presented here may be found in [7].)

#### 3.1 Actions

Suppose  $\mathfrak{U}$  is a HA and  $\mathcal{V}$  a unital associative algebra (both over the same field  $k$ ); a *(left) action* or *generalized (left) derivation* of  $\mathfrak{U}$  on  $\mathcal{V}$  is a linear map  $\triangleright : \mathfrak{U} \otimes \mathcal{V} \rightarrow \mathcal{V}$  satisfying

$$\begin{aligned} (xy) \triangleright v &= x \triangleright (y \triangleright v), & 1 \triangleright v &= v, \\ x \triangleright (vw) &= (x_{(1)} \triangleright v)(x_{(2)} \triangleright w), & x \triangleright 1 &= 1\epsilon(x). \end{aligned} \quad (3.1)$$

for all  $x, y \in \mathfrak{U}$  and  $v, w \in \mathcal{V}$ . (Note that this is merely another way of saying that  $\triangleright$  defines a linear rep of  $\mathfrak{U}$  with right module  $\mathcal{V}$ . A right action  $\triangleleft$  of  $\mathfrak{U}$  on  $\mathcal{V}$  can be defined similarly.)  $\mathfrak{U}$  may therefore be interpreted as an algebra of differential operators acting on elements of  $\mathcal{V}$  from the left, and, as such, may be thought of as providing a means for generalizing infinitesimal transformations.

There are two extremely important examples of such generalized derivations, both of which will be relevant for this work:

- The *(left) adjoint action* of a Hopf algebra  $\mathfrak{U}$  on itself is defined as the linear map  $\overset{\text{ad}}{\triangleright} : \mathfrak{U} \otimes \mathfrak{U} \rightarrow \mathfrak{U}$  given by

$$x \overset{\text{ad}}{\triangleright} y := x_{(1)} y S(x_{(2)}). \quad (3.2)$$

In an undeformed Lie algebra, where  $\Delta(x) = x \otimes 1 + 1 \otimes x$  and  $S(x) = -x$ , this reduces to the usual commutator  $x \overset{\text{ad}}{\triangleright} y \equiv [x, y]$ .

- If  $\mathfrak{U}$  and  $\mathfrak{A}$  are two dually paired Hopf algebras, one may define the (left) action of  $\mathfrak{U}$  on  $\mathfrak{A}$  as

$$x \triangleright a := a_{(1)} \langle x, a_{(2)} \rangle. \quad (3.3)$$

As stated above, this allows the interpretation of  $\mathfrak{U}$  as an algebra of differential operators which act on elements of  $\mathfrak{A}$  (“functions”).

### 3.2 Coactions

$\mathfrak{A}$  is a HA and  $\mathcal{V}$  is again a unital associative algebra; a *(right) coaction* of  $\mathfrak{A}$  on  $\mathcal{V}$  is a linear map  $\Delta_{\mathfrak{A}} : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathfrak{A}$  satisfying

$$(\Delta_{\mathfrak{A}} \otimes \text{id}) \Delta_{\mathfrak{A}}(v) = (\text{id} \otimes \Delta) \Delta_{\mathfrak{A}}(v), \quad (\text{id} \otimes \epsilon) \Delta_{\mathfrak{A}}(v) = v, \quad (3.4)$$

for all  $v \in \mathcal{V}$ . (Many of the following equations will be written using the “Sweedleresque” notation  $\Delta_{\mathfrak{A}}(v) = v^{(1)} \otimes v^{(2)'}$ , where the unprimed elements live in  $\mathcal{V}$ , the primed elements in  $\mathfrak{A}$ .) A left coaction  $\mathfrak{A} \Delta(v) = v^{(1)'} \otimes v^{(2)}$  may be defined similarly. If  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are two HAs which coact on  $\mathcal{V}$  from the left and from the right respectively, it will in general be assumed that they commute with one another, *i.e.*

$$(\mathfrak{A}_1 \Delta \otimes \text{id}) \Delta_{\mathfrak{A}_2}(v) = (\text{id} \otimes \Delta_{\mathfrak{A}_2}) \mathfrak{A}_1 \Delta(v). \quad (3.5)$$

Finally, if  $v$  has the property that  $\Delta_{\mathfrak{A}}(v) = v \otimes 1$ , it is said to be *right-invariant*.

One coaction which will appear later in this work is the *(right) adjoint coaction*  $\Delta^{\text{Ad}} : \mathfrak{A} \rightarrow \mathfrak{A} \otimes \mathfrak{A}$  of  $\mathfrak{A}$  on itself, defined via

$$\Delta^{\text{Ad}}(a) := a_{(2)} \otimes S(a_{(1)})a_{(3)}. \quad (3.6)$$

A comment on terminology: as the reader may have guessed, the reason for the term “coaction” is because of duality. If  $\mathfrak{U}$  and  $\mathfrak{A}$  are dually paired HAs, and  $\triangleright$  is a (left) action of  $\mathfrak{U}$  on some unital associative algebra  $\mathcal{V}$ , then there is a natural way to induce a (right) coaction of  $\mathfrak{A}$  on  $\mathcal{V}$  via

$$v^{(1)} \langle x, v^{(2)'} \rangle = x \triangleright v. \quad (3.7)$$

The interpretation of this particular coaction is straightforward: it is the generalization of a *finite* transformation of an element of  $\mathcal{V}$ , as opposed to the *infinitesimal* transformation provided by the action.

Of particular interest here is the case where  $\mathcal{V} = \mathfrak{U}$ , and the action in question is the adjoint action  $\triangleright^{\text{ad}}$ . Then the coaction  $\Delta_{\mathfrak{A}} : \mathfrak{U} \rightarrow \mathfrak{U} \otimes \mathfrak{A}$  uniquely defined by this pairing, written as  $y \mapsto y^{(1)} \otimes y^{(2)'}$ , is given by

$$x \triangleright^{\text{ad}} y = y^{(1)} \langle x, y^{(2)'} \rangle. \quad (3.8)$$

Notice that this coaction is an algebra map, *i.e.*  $\Delta_{\mathfrak{A}}(xy) = \Delta_{\mathfrak{A}}(x)\Delta_{\mathfrak{A}}(y)$ . This will not in general be the case, as the reader can see from the adjoint coaction  $\Delta^{\text{Ad}}$  above, which is *not* an algebra map from  $\mathfrak{A}$  to  $\mathfrak{A} \otimes \mathfrak{A}$ .

## 4 Quantum Lie Algebras

### 4.1 Basics of Quantum Lie Algebras

Let  $\mathfrak{U}$  be a HA; it is also a *quantum Lie algebra (QLA)* iff there exists a finite-dimensional subspace  $\mathfrak{g} \subset \mathfrak{U}$  ( $\dim \mathfrak{g} = n$ ) which has the following properties<sup>2</sup>:

- (I) as a vector space,  $\mathfrak{U} \equiv U_q(\mathfrak{g})$ , *i.e.* the universal enveloping algebra (UEA) of  $\mathfrak{g}$ , modulo commutation relations;

---

<sup>2</sup>This set of conditions is in effect the dually paired version of the one required for the existence of a general differential calculus [8].

(II) the adjoint action  $\triangleright^{\text{ad}}$  closes on  $\mathfrak{g}$ , *i.e.*  $\mathfrak{U} \triangleright^{\text{ad}} \mathfrak{g} \subseteq \mathfrak{g}$ ;

(III)  $\Delta(x) \in (\mathfrak{g} \otimes 1) \oplus (\mathfrak{U} \otimes \mathfrak{g})$  for all  $x \in \mathfrak{g}$ ;

(IV) for all  $x \in \mathfrak{g}$ ,  $\epsilon(x) = 0$ .

The  $q$  subscript in (I) simply indicates that the commutation relations may be deformed relative to the classical case. (In QG language,  $q$  is the parameter describing the degree of deviation of the QHA from the undeformed case, the latter of which corresponds to  $q = 1$ .)

Let  $\{T_A | A = 1, \dots, n\}$  be a linearly independent basis for  $\mathfrak{g}$  [9]; the above properties require that

$$\Delta(T_A) = T_A \otimes 1 + O_A^B \otimes T_B, \quad \epsilon(T_A) = 0, \quad S(T_A) = -S(O_A^B)T_B, \quad (4.1)$$

where the  $n^2$  elements  $O_A^B \in \mathfrak{U}$  must satisfy

$$\Delta(O_A^B) = O_A^C \otimes O_C^B, \quad \epsilon(O_A^B) = \delta_A^B, \quad S(O_A^B) = (O^{-1})_A^B. \quad (4.2)$$

Condition (II), the closure of  $\mathfrak{g}$  under the adjoint action of  $\mathfrak{U}$ , allows the definition of the  $k$ -numbers  $\hat{\mathbb{R}}^{AB}_{CD}$  and  $f_{AB}^C$  via

$$T_A \triangleright^{\text{ad}} T_B := f_{AB}^C T_C, \quad O_C^B \triangleright^{\text{ad}} T_D := \hat{\mathbb{R}}^{AB}_{CD} T_A \quad (4.3)$$

(from which one can also show  $S^{-1}(O_D^A) \triangleright^{\text{ad}} T_C = (\hat{\mathbb{R}}^{-1})^{AB}_{CD} T_B$ ).  $\hat{\mathbb{R}}$  is referred to as the R-matrix of  $\mathfrak{g}$ , and the  $f$ s are, as in the classical case, just the structure constants. Now, note that the definition of the adjoint action (3.2) implies the identity

$$xy \equiv (x_{(1)} \triangleright^{\text{ad}} y) x_{(2)}; \quad (4.4)$$

therefore, using (4.1–4.3), one can show that

$$\begin{aligned} T_A T_B - \hat{\mathbb{R}}^{CD}_{AB} T_C T_D &= f_{AB}^C T_C, \\ \hat{\mathbb{R}}^{EF}_{AB} O_E^C O_F^D &= O_A^E O_B^F \hat{\mathbb{R}}^{CD}_{EF}, \\ T_A O_B^C - \hat{\mathbb{R}}^{DE}_{AB} O_D^C T_E &= f_{AB}^D O_D^C - O_A^D O_B^E f_{DE}^C, \\ O_A^B T_C &= \hat{\mathbb{R}}^{DE}_{AC} T_D O_E^B. \end{aligned} \quad (4.5)$$

The first of these is the deformation of the classical commutation relation between the generators; the others are due to the difference between the

coproducts of the deformed algebra and the classical versions  $\Delta(T_A) = T_A \otimes 1 + 1 \otimes T_A$ .

The self-consistency of these relations also requires

$$\widehat{\mathbb{R}}_{12}\widehat{\mathbb{R}}_{23}\widehat{\mathbb{R}}_{12} = \widehat{\mathbb{R}}_{23}\widehat{\mathbb{R}}_{12}\widehat{\mathbb{R}}_{23}. \quad (4.6)$$

(Despite the appearance of a very Yang-Baxterlike equation here, the numerical matrix  $\widehat{\mathbb{R}}$  is *not* the rep of a universal R-matrix, which is good, because so far no mention has been made of any quasitriangularity of  $\mathfrak{U}$ .)

## 4.2 The Adjoint Representation

The closure of  $\mathfrak{g}$  under the adjoint action defines the *adjoint rep*  $ad$  of  $\mathfrak{U}$  in the usual way, *i.e.*

$$x \triangleright^{\text{ad}} T_A = ad^B_A(x) T_B. \quad (4.7)$$

This motivates the introduction of an  $n \times n$  matrix  $\mathbb{A}^A_B$  with entries in the dually paired HA  $\mathfrak{A}$ , defined through

$$\langle x, \mathbb{A}^A_B \rangle := ad^A_B(x). \quad (4.8)$$

It follows that

$$f_{AB}{}^C \equiv \langle T_A, \mathbb{A}^C_B \rangle, \quad \widehat{\mathbb{R}}^{AB}{}_{CD} \equiv \langle O_C^B, \mathbb{A}^A_D \rangle. \quad (4.9)$$

Furthermore, by using (3.8), the coaction of  $\mathfrak{A}$  on  $\mathfrak{U}$  is

$$\Delta_{\mathfrak{A}}(T_A) = T_B \otimes \mathbb{A}^B_A. \quad (4.10)$$

The coproduct, counit, antipode, and commutation relations on  $\mathfrak{A}$  have rather familiar forms:

$$\begin{aligned} \Delta(\mathbb{A}^A_B) &= \mathbb{A}^A_C \otimes \mathbb{A}^C_B, & \epsilon(\mathbb{A}^A_B) &= \delta^A_B, \\ S(\mathbb{A}^A_B) &= (\mathbb{A}^{-1})^A_B, & \widehat{\mathbb{R}}\mathbb{A}_1\mathbb{A}_2 &= \mathbb{A}_1\mathbb{A}_2\widehat{\mathbb{R}}, \\ f_{AB}{}^D\mathbb{A}^C_D &= \mathbb{A}^D_A\mathbb{A}^E_B f_{DE}{}^C. \end{aligned} \quad (4.11)$$

The above properties of  $\mathbb{A}$  imply several numerical relations among the R-matrix and structure constants: for instance, if one takes the inner product

of  $\mathbb{A}_N^M$  and the first of (4.5), the deformed version of the Jacobi identity appears:

$$f_{AL}^M f_{BN}^L - \hat{\mathbb{R}}^{CD}{}_{AB} f_{CL}^M f_{DN}^L = f_{AB}^C f_{CN}^M. \quad (4.12)$$

Repeating this for the others just recovers (4.6), as well as

$$\begin{aligned} \hat{\mathbb{R}}^{DC}{}_{BN} f_{AD}^M - \hat{\mathbb{R}}^{DE}{}_{AB} \hat{\mathbb{R}}^{MC}{}_{DF} f_{EN}^F &= \hat{\mathbb{R}}^{MC}{}_{DN} f_{AB}^D - \hat{\mathbb{R}}^{DF}{}_{BN} \hat{\mathbb{R}}^{ME}{}_{AD} f_{EF}^C, \\ \hat{\mathbb{R}}^{MB}{}_{AD} f_{CN}^D &= \hat{\mathbb{R}}^{DE}{}_{AC} \hat{\mathbb{R}}^{FB}{}_{EN} f_{DF}^M. \end{aligned} \quad (4.13)$$

### 4.3 Quasitriangular Quantum Lie Algebras

Given any QHA  $\mathfrak{U}$  with rep  $\rho$ , one can immediately obtain a QLA in the following way [10]: using the matrices  $L^\pm$ , define the matrix  $Y \in M_N(\mathfrak{U})$  [11, 12] by

$$Y := L^+ S(L^-) \equiv (\rho \otimes \text{id})(\mathcal{R}_{21} \mathcal{R}) \equiv \rho(r^\alpha r_\beta) r_\alpha r^\beta. \quad (4.14)$$

As a consequence of (2.18) and (2.19), this matrix satisfies

$$\begin{aligned} \Delta(Y^i{}_j) &= (L^+)^i{}_k S((L^-)^\ell{}_j) \otimes Y^k{}_\ell, & \epsilon(Y^i{}_j) &= \delta_j^i, \\ S(Y^i{}_j) &= S^2((L^-)^k{}_j) S((L^+)^i{}_k), & R_{21} Y_1 R Y_2 &= Y_2 R_{21} Y_1 R. \end{aligned} \quad (4.15)$$

To find the coaction of the dually paired HA  $\mathfrak{A}$  on  $Y$ , there is the following trick [7]: for  $a \in \mathfrak{A}$ , define  $\Upsilon_a \in \mathfrak{U}$  as

$$\Upsilon_a := \langle \mathcal{R}_{21} \mathcal{R}, a \otimes \text{id} \rangle \quad (4.16)$$

Now, note that for  $x \in \mathfrak{U}$ , one may use the fact that  $\mathcal{R}_{21} \mathcal{R}$  is central in  $\Delta(\mathfrak{U})$  to show that

$$x \triangleright^{\text{ad}} \Upsilon_a = \langle x, S(a_{(1)}) a_{(3)} \rangle \Upsilon_{a_{(2)}}. \quad (4.17)$$

Thus, from (3.8),

$$\Delta_{\mathfrak{A}}(\Upsilon_a) = \Upsilon_{a_{(2)}} \otimes S(a_{(1)}) a_{(3)}. \quad (4.18)$$

(Note the appearance of the adjoint coaction (3.6) in this equation.) Since  $Y \equiv \Upsilon_A$ , where  $A$  is the QG matrix, it immediately follows that

$$\Delta_{\mathfrak{A}}(Y^i{}_j) = Y^k{}_\ell \otimes S(A^i{}_k) A^\ell{}_j. \quad (4.19)$$

Furthermore, (4.17) indicates that the adjoint action of  $\mathfrak{U}$  on any element in the subspace  $\{\Upsilon_a | a \in \mathfrak{A}\}$  returns another element of the same subspace. In particular,

$$x \triangleright^{\text{ad}} Y^i_j = \langle x, S(A^i_k) A^\ell_j \rangle Y^k_\ell, \quad (4.20)$$

which is simply a linear combination of the entries of  $Y$ .

Notice that in the classical limit,  $\mathcal{R} \rightarrow 1 \otimes 1$ , so  $Y \rightarrow I1$ ; therefore, define the matrix  $X$  by

$$X := \frac{I1 - Y}{\lambda}, \quad (4.21)$$

where  $\lambda$  is a parameter which vanishes in the classical limit, and describes the deviation of  $Y$  from unity in the deformed case. Using the aforementioned properties of  $Y$ , one immediately finds

$$\begin{aligned} \Delta(X^i_j) &= X^i_j \otimes 1 + (L^+)^i_k S((L^-)^\ell_j) \otimes X^k_\ell, \quad \epsilon(X) = 0, \\ S(X^i_j) &= -S^2((L^-)^\ell_j) S((L^+)^i_k) X^k_\ell, \\ R_{21} X_1 R X_2 - X_2 R_{21} X_1 R &= \frac{1}{\lambda} (R_{21} R X_2 - X_2 R_{21} R), \end{aligned} \quad (4.22)$$

and  $\mathfrak{A}$  coacts on  $X$  as<sup>3</sup>

$$\Delta_{\mathfrak{A}}(X^i_j) = X^k_\ell \otimes S(A^i_k) A^\ell_j. \quad (4.23)$$

Therefore, as was the case with  $Y$ , the adjoint action of  $x \in \mathfrak{U}$  on an entry of  $X$  returns a linear combination of entries of  $X$ , so if  $\mathfrak{g}$  is defined as the subspace of  $\mathfrak{U}$  spanned by the entries of  $X$  over  $k$ , the UEA  $U_q(\mathfrak{g})$  satisfies all four criteria needed for a QLA. To connect to the contents of Section 4.1, one first computes the adjoint action of one entry of  $X$  on another:

$$X^i_j \triangleright^{\text{ad}} X^k_\ell = \frac{1}{\lambda} \left[ \delta^i_j X^k_\ell - (R_{21}^{-1} X_2 R_{21} R)^{im}_{n\ell} \tilde{R}^{nk}_{jm} \right] \quad (4.24)$$

(see (A.3) for the definition of  $\tilde{R}$ ). Now, comparison of (4.24) with (4.5) immediately leads to the identifications

$$T_{(ij)} \equiv X^i_j, \quad O_{(ij)}^{(k\ell)} \equiv (L^+)^i_k S((L^-)^\ell_j), \quad \mathbb{A}^{(ij)}_{(k\ell)} \equiv S(A^k_i) A^j_\ell, \quad (4.25)$$

---

<sup>3</sup>Although this work is concerned primarily with the *right* coaction of the QG  $\mathfrak{A}$  on  $\mathfrak{U}$ , there is also a *left* coaction  $x \mapsto 1 \otimes x$ , *i.e.*  $\mathfrak{U}$  is left-invariant. The construction presently being discussed reproduces this, and motivates calling the resulting QLA “bicovariant” [7, 8].



and

$$\begin{aligned}\widehat{\mathbb{R}}^{(ab)(cd)}_{(ij)(k\ell)} &= \tilde{R}^{mk}_{jn}\widehat{R}^{sd}_{m\ell}(\widehat{R}^{-1})^{ni}_{ra}\widehat{R}^{rb}_{sc}, \\ f_{(ij)(k\ell)}^{(rs)} &= \frac{1}{\lambda} \left[ \delta^i_j \delta^k_r \delta^s_\ell - \tilde{R}^{mk}_{jn}(\widehat{R}^{-1})^{ni}_{tr}(\widehat{R}^2)^{ts}_{m\ell} \right].\end{aligned}\quad (4.26)$$

There are three important sum relations which follow from these last two identifications:

$$f_{AB}{}^C I_C = 0, \quad \widehat{\mathbb{R}}^{CD}{}_{AB} I_C = \delta^D_A I_B, \quad \widehat{\mathbb{R}}^{CD}{}_{AB} I_D = \delta^C_B I_A - \lambda f_{AB}{}^C, \quad (4.27)$$

where, for the sake of simplicity, the quantity  $I_{(ij)} := \delta^i_j$  has been introduced. There is also the additional relation  $I_B \mathbb{A}^B{}_A = I_A 1$ .

(To explain an earlier parenthetical comment, I should point out that the numerical R-matrix in this rep, *i.e.*  $\mathbb{R}^{AB}{}_{CD} := \langle \mathcal{R}, \mathbb{A}^A{}_C \otimes \mathbb{A}^B{}_D \rangle$ , is

$$\mathbb{R}^{(ab)(cd)}_{(ij)(k\ell)} = \tilde{R}^{mk}_{jn}\widehat{R}^{sb}_{m\ell}\widehat{R}^{ni}_{rc}(\widehat{R}^{-1})^{rd}_{sa}, \quad (4.28)$$

which is *not* equal to  $\widehat{\mathbb{R}}^{(cd)(ab)}_{(ij)(k\ell)}.$

## 5 The Killing Form

### 5.1 The Killing Form for a Quasitriangular Hopf Algebra

Recall that for a Lie algebra  $\mathfrak{g}$  over a field  $k$ , a Killing form  $\eta$  is defined to be a symmetric invariant linear map from  $\mathfrak{g} \otimes \mathfrak{g}$  to  $k$ , *i.e.*  $\eta(x \otimes y) = \eta(y \otimes x)$  and  $\eta([z, x] \otimes y) + \eta(x \otimes [z, y]) = 0$  (*cf.* [13]). (Such a form always exists for finite-dimensional Lie algebras: one needs only take the trace of two elements in the adjoint rep of  $\mathfrak{g}$ .) Now, with all the formalism developed so far, it is finally possible to define an object on a QHA which generalizes these required properties, in the following way:

**Definition** Let  $\mathfrak{U}$  be a QHA with universal R-matrix  $\mathcal{R}$  and rep  $\rho$ . The linear map  $\eta^{(\rho)} : \mathfrak{U} \otimes \mathfrak{U} \rightarrow k$ , the *Killing form associated with  $\rho$* , is given by

$$\eta^{(\rho)}(x \otimes y) := \text{tr}_\rho(uxy) \quad (5.1)$$

for  $x, y \in \mathfrak{U}^4$ .

$\eta^{(\rho)}$  defined in this way has the following properties:

$$\begin{aligned}\eta^{(\rho)}(y \otimes x) &= \eta^{(\rho)}(x \otimes S^2(y)), \\ \eta^{(\rho)}\left((z_{(1)}^{\text{ad}} x) \otimes (z_{(2)}^{\text{ad}} y)\right) &= \eta^{(\rho)}(x \otimes y) \epsilon(z),\end{aligned}\tag{5.2}$$

for all  $x, y, z \in \mathfrak{U}$ . The first of these is immediate from the properties of  $u$  and the cyclicity of the trace:

$$\begin{aligned}\eta^{(\rho)}(y \otimes x) &= \text{tr}_\rho(uyx) \\ &= \text{tr}_\rho((uyu^{-1})ux) \\ &= \text{tr}_\rho(ux(uyu^{-1})) \\ &= \eta^{(\rho)}(x \otimes S^2(y)).\end{aligned}\tag{5.3}$$

The second is obtained by noting that

$$\begin{aligned}\text{tr}_\rho\left(u(z^{\text{ad}} x)\right) &= \text{tr}_\rho\left(uz_{(1)}xS(z_{(2)})\right) \\ &= \eta^{(\rho)}\left(z_{(1)} \otimes xS(z_{(2)})\right) \\ &= \eta^{(\rho)}\left(xS(z_{(2)}) \otimes S^2(z_{(1)})\right) \\ &= \text{tr}_\rho\left(uxS(S(z_{(1)})z_{(2)})\right) \\ &= \text{tr}_\rho(ux) \epsilon(z),\end{aligned}\tag{5.4}$$

so that by replacing  $x$  by  $xy$  above, the result follows.

The first of (5.2) is obviously a generalization of the “symmetry” of the Killing form, since  $S^2 = \text{id}$  in the classical case; since the adjoint action  $^{\text{ad}}$  is the HA generalization of the classical commutator, the second expresses the desired invariance property of  $\eta^{(\rho)}$ . In fact, this identification may be made more obvious by realizing that the latter may be rewritten as

$$\eta^{(\rho)}\left((z^{\text{ad}} x) \otimes y\right) + \eta^{(\rho)}\left(x \otimes ((-S(z))^{\text{ad}} y)\right) = 0.\tag{5.5}$$

---

<sup>4</sup>This differs from the definition in [15] by an overall antipode, *i.e.*  $\eta^{(\rho)}(x \otimes y) := \text{tr}_\rho(S(uxy))$ , which corresponds to taking the trace over the contragredient rep.

The invariance under the adjoint action may be thought of as how the Killing form behaves under an “infinitesimal” transformation; however, recall that  $\triangleright^{\text{ad}}$  also induces the “finite” transformation (3.8). Using this fact, one sees that the left-hand side of the second of (5.2) becomes

$$\begin{aligned}\eta^{(\rho)} \left( (z_{(1)} \triangleright^{\text{ad}} x) \otimes (z_{(2)} \triangleright^{\text{ad}} y) \right) &= \eta^{(\rho)} \left( x^{(1)} \otimes y^{(1)} \right) \langle z_{(1)}, x^{(2)'} \rangle \langle z_{(2)}, y^{(2)'} \rangle \\ &= \eta^{(\rho)} \left( x^{(1)} \otimes y^{(1)} \right) \langle \Delta(z), x^{(2)'} \otimes y^{(2)'} \rangle \\ &= \eta^{(\rho)} \left( x^{(1)} \otimes y^{(1)} \right) \langle z, x^{(2)'} y^{(2)'} \rangle.\end{aligned}\quad (5.6)$$

The right-hand side is equivalent to  $\eta^{(\rho)}(x \otimes y) \langle z, 1 \rangle$ , so since these must agree for all  $z \in \mathfrak{U}$ ,

$$\eta^{(\rho)} \left( x^{(1)} \otimes y^{(1)} \right) x^{(2)'} y^{(2)'} = \eta^{(\rho)}(x \otimes y) 1. \quad (5.7)$$

which expresses the “finite” invariance of the Killing form.

## 5.2 The Killing Metric for a Quantum Lie Algebra

In the case when  $\mathfrak{U}$  is not only a QHA but also a QLA with generators  $\{T_A | A = 1, \dots, n\}$ , one can define the *Killing metric* associated with the rep  $\rho$  as

$$\eta_{AB}^{(\rho)} := \eta^{(\rho)}(T_A \otimes T_B). \quad (5.8)$$

It is now convenient to introduce the quantities  $\tau_A(\rho) := \text{tr}_\rho(uT_A)$ ; one can deduce from the results of the previous sections that these “deformed traces” must satisfy

$$\begin{aligned}\text{tr}_\rho \left( u(T_A \triangleright^{\text{ad}} T_B) \right) &= f_{AB}^C \tau_C(\rho) = 0, \\ \text{tr}_\rho \left( u(O_A \triangleright^{\text{ad}} T_C) \right) &= \hat{\mathbb{R}}^{DB}_{AC} \tau_D(\rho) = \delta_A^B \tau_C(\rho), \\ \text{tr}_\rho(uT_B) \mathbb{A}^B_A &= \tau_B(\rho) \mathbb{A}^B_A = \tau_A(\rho) 1.\end{aligned}\quad (5.9)$$

The first of (5.9) implies that by multiplying (4.5) by  $u$  and tracing over  $\rho$ ,  $\eta_{AB}^{(\rho)}$  satisfies

$$\eta_{AB}^{(\rho)} = \hat{\mathbb{R}}^{CD}_{AB} \eta_{CD}^{(\rho)} = \mathbb{D}^C_A \eta_{BC}^{(\rho)} \quad (5.10)$$

(see (A.17) for the definition of  $\mathbb{D}$ ), which expresses the “symmetry” of the Killing metric. One can also obtain the “total antisymmetry” of the structure constants in a similar way; since the counits of all the generators vanish, (5.2) requires that

$$\begin{aligned}
0 &= \eta^{(\rho)} \left( (T_{C(1)} \rhd^{\text{ad}} T_A) \otimes (T_{C(2)} \rhd^{\text{ad}} T_B) \right) \\
&= \eta^{(\rho)} \left( (T_C \rhd^{\text{ad}} T_A) \otimes T_B \right) + \eta^{(\rho)} \left( (O_C \rhd^{\text{ad}} T_A) \otimes (T_D \rhd^{\text{ad}} T_B) \right) \\
&= f_{CA}{}^D \eta^{(\rho)} (T_D \otimes T_B) + \widehat{\mathbb{R}}^{ED} {}_C A f_{DB}{}^F \eta^{(\rho)} (T_E \otimes T_F). \tag{5.11}
\end{aligned}$$

Therefore,

$$f_{CA}{}^D \eta_{DB}^{(\rho)} + \widehat{\mathbb{R}}^{ED} {}_C A f_{DB}{}^F \eta_{EF}^{(\rho)} = 0. \tag{5.12}$$

By using (4.10), together with (5.7), the invariance of the Killing metric under finite transformations takes the form

$$\eta_{CD}^{(\rho)} \mathbb{A}^C{}_A \mathbb{A}^D{}_B = \eta_{AB}^{(\rho)} 1. \tag{5.13}$$

### 5.3 Quadratic Casimirs

Now, suppose that  $\eta_{AB}^{(\rho)}$  is invertible, *i.e.* there exists a numerical matrix  $\eta^{(\rho)AB}$  such that

$$\eta_{AC}^{(\rho)} \eta^{(\rho)CB} = \eta^{(\rho)BC} \eta_{CA}^{(\rho)} = \delta_A^B. \tag{5.14}$$

Then (5.13) implies that  $\mathbb{A}^A{}_C \mathbb{A}^B{}_D \eta^{(\rho)CD} = \eta^{(\rho)AB} 1$ ; this in turn indicates that the *quadratic casimir* defined by

$$Q^{(\rho)} := \eta^{(\rho)AB} T_A T_B \tag{5.15}$$

is central. Why? First, note that  $Q^{(\rho)}$  is right-invariant:

$$\begin{aligned}
\Delta_{\mathfrak{A}}(Q^{(\rho)}) &= \eta^{(\rho)AB} \Delta_{\mathfrak{A}}(T_A) \Delta_{\mathfrak{A}}(T_B) \\
&= T_C T_D \otimes \eta^{(\rho)AB} \mathbb{A}^C{}_A \mathbb{A}^D{}_B \\
&= T_C T_D \eta^{(\rho)CD} \otimes 1 \\
&= Q^{(\rho)} \otimes 1. \tag{5.16}
\end{aligned}$$

Now, recall (3.8) and (4.4); the first of these states that if  $y \in \mathfrak{U}$  is right-invariant,  $x \rhd^{\text{ad}} y = \epsilon(x)y$  for all  $x$ . The second therefore implies

$$xy = (x_{(1)} \rhd^{\text{ad}} y) x_{(2)} = \epsilon(x_{(1)}) y x_{(2)} = yx, \tag{5.17}$$

namely, *any* right-invariant element of  $\mathfrak{U}$  is central. Since the right-invariance of  $Q^{(\rho)}$  has just been demonstrated, it follows that the quadratic casimir commutes with everything in the algebra, just as in the classical case.

## 5.4 The Canonical Killing Metric

Suppose that there exists at least one rep  $\rho_0$  for which the associated Killing metric  $\eta_{AB}^{(0)}$  is invertible, *i.e.*  $\eta^{(0)AB}$  exists. Given another rep  $\rho$ , define the numerical matrix  $K^A{}_B(\rho)$  by

$$K^A{}_B(\rho) := \eta^{(0)AC} \eta_{CB}^{(\rho)}, \quad (5.18)$$

so that  $\eta_{AB}^{(\rho)} \equiv \eta_{AC}^{(0)} K^C{}_B(\rho)$ . Therefore, (5.12) may be rewritten in terms of  $\eta^{(0)}$  and  $K(\rho)$ :

$$\begin{aligned} 0 &= f_{CA}{}^D \left( \eta_{DF}^{(0)} K^F{}_B(\rho) \right) + \hat{\mathbb{R}}^{ED} {}_CA f_{DB}{}^F \left( \eta_{EM}^{(0)} K^M{}_F(\rho) \right) \\ &= - \left( \hat{\mathbb{R}}^{ED} {}_CA f_{DF}{}^M \eta_{EM}^{(0)} \right) K^F{}_B(\rho) + \hat{\mathbb{R}}^{ED} {}_CA f_{DB}{}^F \eta_{EM}^{(0)} K^M{}_F(\rho) \\ &= \hat{\mathbb{R}}^{ED} {}_CA \eta_{EM}^{(0)} \left( K^M{}_F(\rho) f_{DB}{}^F - f_{DF}{}^M K^F{}_B(\rho) \right). \end{aligned} \quad (5.19)$$

Since both  $\hat{\mathbb{R}}$  and  $\eta^{(0)}$  are invertible, this may be multiplied by  $(\hat{\mathbb{R}}^{-1})^{CA} {}_{PN} \eta^{(0)LP}$  to obtain the matrix equation

$$[K(\rho), ad(T_N)]^L{}_B = 0, \quad (5.20)$$

*i.e.* the matrix  $K(\rho)$  commutes with all the generators in the adjoint rep. Therefore, if the adjoint rep is irreducible, Schur's lemma can be applied, which would indicate that  $K(\rho)$  is proportional to the identity matrix. This in turn gives the result that  $\eta_{AB}^{(\rho)} \propto \eta_{AB}^{(0)}$  for *all*  $\rho$ , so that, up to a multiplicative constant, all Killing metrics may be written in terms of a canonical, *i.e.* rep-independent, Killing metric  $\eta_{AB}$ . Once the normalization of  $\eta$  is fixed, this allows the definition of the *index*  $\mathcal{I}(\rho)$  of the rep  $\rho$  in exact analogy with the classical case, *i.e.*  $\eta_{AB}^{(\rho)} \equiv \mathcal{I}(\rho) \eta_{AB}$ .

The existence of at least one rep where the Killing metric is invertible has been assumed in the above proof, so since all metrics are proportional to  $\eta_{AB}$ , the canonical Killing metric itself must be invertible. This fact implies the existence of a rep-independent quadratic casimir  $Q := \eta^{AB} T_A T_B$ , which of course will still be central in the QLA. Therefore, as in the classical case, for any rep  $\rho$ ,  $Q \equiv \mathcal{I}(\rho) Q^{(\rho)}$ .

## 5.5 Comments on the Adjoint Representation

Before proceeding to the particular case where the QLA of interest is  $U_q(\mathfrak{su}(N))$ , it is necessary to consider some of the consequences of the trace relations (5.9), specifically as pertains to the adjoint rep. These relations are obviously trivially satisfied for  $\tau_A(\rho) \equiv 0$ , which is the case for the classical compact Lie algebras, where the generators are traceless in all reps. However, there is no reason to assume that this tracelessness condition still holds when the object in question is a QLA (or even a classical *noncompact* Lie algebra, for that matter), *i.e.* there may exist a rep  $\rho$  such that  $\tau_A(\rho)$  does not vanish. If this is in fact the case, one is then able to deduce the existence of another numerical object  $\mathcal{D}^A$  which satisfies

$$f_{AB}{}^C \mathcal{D}^B = 0. \quad (5.21)$$

Why should this quantity exist? From the last of (5.9),  $\tau_A(\rho) 1$  is an algebra-valued eigenvector of  $\mathbb{A}^t$  with eigenvalue unity. The transpose of any matrix has the same eigenvalues as the original, so this implies the existence of a numerical quantity  $\mathcal{D}^A$  such that  $\mathcal{D}^A 1$  is a nonzero algebra-valued eigenvector of  $\mathbb{A}$  with unit eigenvalue, *i.e.*

$$\mathbb{A}^A{}_B \mathcal{D}^B = \mathcal{D}^A 1. \quad (5.22)$$

This in turn implies (5.21), as well as

$$\widehat{\mathbb{R}}^{CA}{}_{BD} \mathcal{D}^D = \delta_B^A \mathcal{D}^C. \quad (5.23)$$

(4.10) and (5.22) together give the important result that the quantity

$$T_0 := \mathcal{D}^A T_A \quad (5.24)$$

is right-invariant, and therefore commutes with the entire algebra. Now, assume that the deformed trace of this element  $\tau_0(\rho) := \text{tr}_\rho(u T_0)$  vanishes for all reps. Therefore, if  $\rho$  is an irrep of  $\mathfrak{U}$ , then Schur's lemma states that there exists a  $k$ -number  $\mu(\rho)$  such that  $\rho(T_0) = \mu(\rho) I$ , where  $I$  is the identity matrix in the corresponding irrep. Thus,  $\tau_0(\rho) = \mu(\rho) \text{tr}_\rho(u)$ . The quantity  $\text{tr}_\rho(u)$  may be thought of as the “deformed dimension” of the irrep, and it is assumed here that it is always nonvanishing; hence, the condition  $\tau_0(\rho) = 0$  implies that  $\mu(\rho)$  vanishes. However, if this holds for *all* irreps, then  $T_0$  must

be identically zero as an element of  $\mathfrak{U}$ . The linear independence of the  $n$  generators  $\{T_A\}$  then leads to the conclusion that  $\mathcal{D}^A \equiv 0$ , contrary to the initial assumption, namely that there exists a nonvanishing  $\mathcal{D}^A$ . Therefore, there must exist a rep  $\rho'$  for which  $\tau'_0 := \tau_0(\rho') \neq 0$ , and hence the individual traces  $\tau'_A := \tau_A(\rho')$  are also nonvanishing.

The existence of this particular rep allows the definition of a new set of generators  $\{T'_A | A = 1, \dots, n\}$  as

$$T'_A := T_A - \frac{\tau'_A}{\tau'_0} T_0. \quad (5.25)$$

(Note that these are traceless in the rep  $\rho'$ .) The commutation relations and adjoint actions now take the forms

$$T'_A T'_B - \hat{\mathbb{R}}^{CD}{}_{AB} T'_C T'_D = \left[ f_{AB}{}^C - (\delta_A^D \delta_B^C - \hat{\mathbb{R}}^{CD}{}_{AB}) \frac{\tau'_D}{\tau'_0} T_0 \right] T'_C \quad (5.26)$$

and

$$\begin{aligned} T_0^{\text{ad}} T_0 &= 0, & T'_A{}^{\text{ad}} T_0 &= 0, \\ T_0^{\text{ad}} T'_A &= \mathcal{D}^B f_{BA}{}^C T'_C, & T'_A{}^{\text{ad}} T'_B &= \left( \delta_A^D - \frac{\tau'_A}{\tau'_0} \mathcal{D}^D \right) f_{DB}{}^C T'_C, \end{aligned} \quad (5.27)$$

However, the primed generators are *not* linearly independent, because  $\mathcal{D}^A T'_A \equiv 0$ , so they do not form a basis for  $\mathfrak{g}$ . However, if one of them, for instance  $T'_n$ , is replaced by  $T_0$ , then  $\{T_0, T'_a | a = 1, \dots, n-1\}$  is a linearly independent collection of  $n$  vectors, and therefore is a basis for  $\mathfrak{g}$ . (For the rest of this work, when a capital index  $A$  appears on a primed generator, it will take the values  $A = 0, 1, \dots, n-1$ , with  $T'_0 = T_0$ , and a small index  $a$  the values  $a = 1, \dots, n-1$ .)

The structure constants  $f'$  in this new basis, and thus the matrices in the adjoint rep, are easily obtained from (5.27), and the reader can immediately see that

$$f'^0{}_0 = f'^a{}_0 = f'^0{}_a = f'^b{}_a = f'^0{}_{ab} = f'^a{}_{ab} = 0, \quad (5.28)$$

*i.e.* the only nonvanishing structure constants are  $f'_{Aa}{}^b$ . In this basis all the matrices in the adjoint rep will be block-diagonal, with a zero in the upper left-hand corner and an  $(n-1) \times (n-1)$  matrix as the lower right-hand

submatrix (corresponding to the  $(00)$  and  $(ab)$  entries respectively). This set of matrices closes under multiplication, so the  $(n-1) \times (n-1)$  matrices

$$ad'^a{}_b(T'_A) := f'^a{}_{Ab} \quad (5.29)$$

form a rep of the QLA.

## 6 The Quantum Lie Algebra $U_q(\mathfrak{su}(N))$

The discussions and conclusions just presented in Section 5 closely parallel those usually found when talking about undeformed compact Lie algebras, *e.g.* the existence of a rep for which the associated Killing metric is invertible, irreducibility of the adjoint rep, tracelessness of the basis, *etc.* For the classical compact Lie algebras, all these assumptions hold: the Killing metric in the adjoint rep is positive definite (this is actually the *definition* of a compact Lie algebra) and thus invertible, the adjoint rep is an irrep, and all the generators are traceless (the “ $S$ ” in  $SU(N)$ ). Thus, results like the existence of  $\eta_{AB}$  and a rep-independent quadratic casimir follow. However, these assumptions may not be valid for a *deformed* Lie algebra; hence, to use any of the conclusions of the previous section, one must first ask how many of these still hold for a QLA.

Because the classical Lie algebra  $\mathfrak{su}(N)$  is the one which is most familiar to most physicists, the deformed version  $U_q(\mathfrak{su}(N))$  is the best example with which to illustrate many of the results just obtained. This Section will examine this particular QLA in detail.

### 6.1 R-Matrix Construction

As shown in Section 4.3, if a numerical R-matrix for a QHA is given, a QLA may be constructed;  $U_q(\mathfrak{su}(N))$  is the QLA found by this method using the numerical R-matrix for the QG  $SU_q(N)$ , which in the fundamental rep is given by multiplying the R-matrix for  $A_{N-1}$  in [6] by  $q^{-\frac{1}{N}}$  [14]:

$$R = q^{-\frac{1}{N}} \left( q \sum_I E_{II} \otimes E_{II} + \sum_{I \neq J} E_{II} \otimes E_{JJ} + \lambda \sum_{I > J} E_{IJ} \otimes E_{JI} \right), \quad (6.1)$$

where  $E_{IJ}$  is the  $N \times N$  numerical matrix whose only nonzero entry is a 1 at  $(I, J)$ , the tensor product which appears is that between numerical



spaces, and  $k = \mathbb{R}$ . ( $\lambda := q - q^{-1}$  is the parameter referred to earlier which describes how the QG differs from the classical group as a function of  $q$ .) The normalization for the matrix  $D$  is chosen so that  $D = \text{diag}(1, q^2, \dots, q^{2(N-1)})$ , which fixes the constants  $\alpha$  and  $\beta$  from the Appendix to be  $q^{2N-1-\frac{1}{N}}$  and  $q^{1-\frac{1}{N}}$  respectively.

The commutation relations (4.22) and adjoint actions (4.24) follow, with the structure constants given by (4.26). Now that they are known explicitly in terms of the above numerical R-matrix, one can immediately obtain several results specific to  $U_q(\mathfrak{su}(N))$ : first of all, suppose that  $V_{(ij)} := V^i_j$  is a null eigenvector of the transposed adjoint matrices, *i.e.*

$$f_{(ij)(k\ell)}^{(rs)} V_{(rs)} \equiv \frac{1}{\lambda} \tilde{R}^{mk}_{jn} (\hat{R} V_2 - \hat{R}^{-1} V_2 \hat{R}^2)^{ni}_{m\ell} = 0. \quad (6.2)$$

From this, it follows that  $V_2$  must commute with  $\hat{R}^2$ . This implies that  $V$  must be a multiple of the identity matrix: to see why, suppose  $V_2$  commutes with a matrix  $\hat{M}$  such that  $\hat{M}$  exists, where  $M^{ij}_{k\ell} := \hat{M}^{ji}_{k\ell}$ . If this is the case, then

$$\begin{aligned} 0 &= \tilde{M}^{mj}_{kn} (\hat{M} V_2 - V_2 \hat{M})^{ni}_{m\ell} \\ &= \tilde{M}^{mj}_{kn} M^{in}_{mr} V^r_{\ell} - \tilde{M}^{mj}_{kn} V^i_s M^{sn}_{m\ell} \\ &= \delta^i_k V^j_{\ell} - V^i_k \delta^j_{\ell}, \end{aligned} \quad (6.3)$$

which can only be satisfied if  $V^i_j \propto \delta^i_j$ . The numerical R-matrix for  $SU_q(N)$  satisfies the quadratic characteristic equation  $\hat{R}^2 - q^{-\frac{1}{N}} \lambda \hat{R} - q^{-\frac{2}{N}} I = 0$ , so  $V_2$  commutes with  $\hat{R}$ . Since  $\tilde{R}$  exists, the above result applies, and all null eigenvectors of the transposed adjoint matrices must be proportional to the identity. This implies that since the traces  $\tau_{(ij)}(\rho)$  are such eigenvectors, all of them are equal to a rep-dependent constant times the identity. (From (4.27), it is easily seen that these also satisfy the first two of (5.9) as well.) In fact, for the fundamental rep  $fn$ , in which the matrices take the form

$$fn^i_j(T_{(k\ell)}) = \frac{1}{\lambda} (I - \hat{R}^2)^{ik}_{j\ell}, \quad (6.4)$$

a quick calculation gives

$$\tau_{(ij)}(fn) = q^{-\frac{1}{N}} \left( \left[ \frac{1}{N} \right]_q [N]_{q^{-1}} - 1 \right) \delta^i_j \quad (6.5)$$

(where I have used the standard notation for the “quantum number”,

$$[m]_q := \frac{q^{2m} - 1}{q^2 - 1}, \quad (6.6)$$

so called because as  $q \rightarrow 1$ ,  $[m]_q \rightarrow m$ ). In the classical limit, this vanishes for all  $N$ . This should come as no surprise: after all, the adjoint rep of the classical Lie algebra  $\mathfrak{su}(N)$  is irreducible, so there cannot exist any common nonzero null eigenvector for the matrices  $ad(T_A)^t$ . However, for  $q \neq 1$ , this is nonvanishing, and so it implies the existence of  $T_0$  and will allow the construction of the  $N^2 - 1$  primed generators, which together with  $T_0$  give a basis for  $\mathfrak{g}$ . The same sort of argument allows one to determine that all matrices  $W^{(ij)} := W^j_i$  satisfying  $f_{(ij)(k\ell)}^{(rs)} W^{(k\ell)} = 0$  must be proportional to  $(D^{-1})^j_i$ . From (5.21), this includes  $\mathcal{D}^A$ , so  $\mathcal{D}^{(ij)} \propto (D^{-1})^j_i$ .

The arguments just given have two immediate consequences: first of all, assume there exists a vector  $\vec{e} = (e^1, \dots, e^{n-1})$  which satisfies  $ad'(T'_A) \cdot \vec{e} = \vec{0}$ ; since each  $ad(T'_A)$  is block-diagonal with a zero as its (00) entry, then any vector of the form  $(\nu, \vec{e})$ ,  $\nu$  being some constant, will be a null eigenvector of  $ad(T'_A)$ . However, it has just been shown that all such eigenvectors are proportional to  $D^{-1}$ , and thus to  $\mathcal{D}^A$ . In the primed basis,  $\mathcal{D}^0 = 1$  and  $\mathcal{D}^a = 0$ , so any vector annihilated by all the adjoint matrices must be a multiple of  $(1, \vec{0})$ . This implies that  $\vec{e}$  must vanish, and that there are *no* shared nonzero null eigenvectors of the matrices  $ad'(T'_A)$ . One therefore concludes that the  $(N^2 - 1)$ -dimensional rep of  $U_q(\mathfrak{su}(N))$  given by  $ad'$  is in fact an irrep, and  $ad'(T_0)$ , being central, will be proportional to the identity matrix.

Second, notice that

$$\frac{\tau_{(ij)}(\rho)}{\tau_0(\rho)} \mathcal{D}^{(k\ell)} = \frac{1}{\text{tr}(D^{-1})} \delta_j^i (D^{-1})^\ell_k, \quad (6.7)$$

which is *independent* of the constants of proportionality for both  $\tau_{(ij)}(\rho)$  and  $\mathcal{D}^{(ij)}$ . The generators  $T_0$  and  $T'_{(ij)}$  therefore take the forms

$$T_0 = \text{tr}(D^{-1}X), \quad T'_{(ij)} = X^i_j - \frac{1}{\text{tr}(D^{-1})} \text{tr}(D^{-1}X) \delta_j^i. \quad (6.8)$$

(The above expression for  $T_0$  together with (A.14) provides further proof that  $T_0$  is right-invariant, and thus central.) A consequence of the re-independence of (6.7) is that the primed generators are traceless in *all* reps,

not just the one for which  $\tau'_0$  is nonvanishing (*e.g.*  $fn$ ). Thus, when one of them is replaced by  $T_0$  to obtain a linearly independent basis for  $\mathfrak{g}$ , the result is that the QLA is generated by the  $(N^2 - 1)$ -dimensional subspace of traceless basis elements  $\mathfrak{g}' = \{T'_a\}$  and a central element  $T_0$ . In fact, this also implies that  $T_0$  must vanish in the classical limit, because  $\mathfrak{g}' = \mathfrak{su}(N)$  in this case, and one requires that  $U_q(\mathfrak{su}(N)) \rightarrow U(\mathfrak{su}(N))$ .

$U_q(\mathfrak{su}(N))$  supposedly describes a deformation of a unitary algebra, so a hermiticity condition on its elements must be imposed: this is given in terms of the involution acting on the matrices  $L^\pm$  as  $\theta(L^\pm) := S(L^\mp)^t$ , which implies that the generator matrix is hermitian, *i.e.*  $\theta(X) = X^t$ , or  $\theta(T_{(ij)}) = T_{(ji)}$ . If the subspace  $\mathfrak{g}$  is to be a vector space over  $\mathbb{R}$  as in the classical case, then for  $x \in \mathfrak{g}$ ,  $\theta(x) = x$ . Being just a linear combination of the generators, such an element  $x$  may therefore be written

$$x = \xi^A T_A = \text{tr}(\Xi X), \quad (6.9)$$

where  $\xi^{(ij)} \equiv \Xi^j_i$  are the numerical coordinates of  $x$  in the basis  $\{T_{(ij)}\}$ . For  $x$  to be self-adjoint, the matrix  $\Xi$  must be hermitian.

## 6.2 Properties of the Killing Metric

In the fundamental rep  $fn$ , the generators are given by (6.4), and  $u$  by  $q^{1+\frac{1}{N}-2N}D$  (see Appendix), so the Killing metric associated with the fundamental rep in the basis  $\{T_{(ij)}\}$  may be found explicitly:

$$\begin{aligned} \eta_{(ij)(k\ell)}^{(fn)} &= q^{-\frac{1}{N}} \left( q \left[ 1 - \frac{1}{N} \right]_q - \frac{1}{q} \left[ 1 + \frac{1}{N} \right]_{q^{-1}} + q^{\frac{2}{N}-3} \left[ \frac{1}{N} \right]_{q^{-1}}^2 [N]_{q^{-1}} \right) \delta_j^i \delta_\ell^k \\ &\quad + q^{1-\frac{3}{N}-2N} \delta_\ell^i D^k_j. \end{aligned} \quad (6.10)$$

Now, suppose one switches to the primed basis and computes the Killing metric for a rep  $\rho$  which reduces to irreps  $\{\rho_i\}$ . Then

$$\begin{aligned} \eta_{0a}^{(\rho)} &= \text{tr}_\rho(u T_0 T'_a) \\ &= \sum_i \text{tr}_{\rho_i}(u T_0 T'_a) \\ &= \sum_i \mu(\rho_i) \text{tr}_{\rho_i}(u T'_a), \end{aligned} \quad (6.11)$$

which vanishes due to the tracelessness of the primed generators in all reps. Furthermore, since  $T_0$  is central, then  $\eta_{a0}^{(\rho)} \equiv \eta_{0a}^{(\rho)}$ , so the Killing metric will be a block-diagonal matrix with  $\eta_{00}^{(\rho)}$  in the upper left-hand corner and the  $(N^2 - 1) \times (N^2 - 1)$  matrix  $\eta_{ab}^{(\rho)}$  in the lower right-hand corner.

Any element  $x \in \mathfrak{g}$  may be written either as in (6.9), or in terms of the primed basis as  $x = \xi^0 T_0 + \xi^a T'_a$ , from which it follows from (6.8) that  $\xi^0 = \frac{\text{tr} \Xi}{\text{tr} D^{-1}}$ . As the reader has just seen, the Killing metric is block-diagonal in the primed basis, so  $\eta_{AB}^{(fn)} \xi^A \xi^B = \eta_{00}^{(fn)} (\xi^0)^2 + \eta_{ab}^{(fn)} \xi^a \xi^b$ . Therefore, by using (6.10),

$$\begin{aligned}
\eta_{ab}^{(fn)} \xi^a \xi^b &= \eta_{AB}^{(fn)} \xi^A \xi^B - \eta_{00}^{(fn)} (\xi^0)^2 \\
&= \eta_{(ij)(k\ell)}^{(fn)} \left[ \Xi^j_i \Xi^\ell_k - (D^{-1})^j_i (D^{-1})^\ell_k \left( \frac{\text{tr} \Xi}{\text{tr} D^{-1}} \right)^2 \right] \\
&= q^{1-\frac{3}{N}-2N} \left[ \text{tr}(D \Xi^2) - \frac{(\text{tr} \Xi)^2}{\text{tr} D^{-1}} \right] \\
&= q^{1-\frac{3}{N}-2N} \text{tr} \left[ D \left( \Xi - \frac{\text{tr} \Xi}{\text{tr} D^{-1}} D^{-1} \right)^2 \right] \\
&= q^{1-\frac{3}{N}-2N} \text{tr} \left| D^{\frac{1}{2}} \Xi - \frac{\text{tr} \Xi}{\text{tr} D^{-1}} D^{-\frac{1}{2}} \right|^2, \tag{6.12}
\end{aligned}$$

where the fact that  $D$  is hermitian and positive definite (recall that there is a basis where it is diagonal with each entry being the square of a real number) has been used. The quantity above is thus always nonnegative, and vanishes only when  $\Xi \propto D^{-1}$ , *i.e.*  $x \propto T_0$  and  $\xi^a = 0$ . The Killing metric  $\eta_{ab}^{(fn)}$  constructed using only the primed generators is positive definite, and thus invertible.

Therefore, the matrix  $K(\rho)$  from (5.18) can be constructed by using  $\rho_0 = fn$ , and it too will be block-diagonal. In particular, its upper left-hand entry will be  $\eta^{(fn)00} \eta_{00}^{(\rho)}$ . However, when  $K(\rho)$  is then multiplied by any matrix in the adjoint rep, the result will have a zero in this spot, since all adjoint matrices have a vanishing entry there. The commutation relation (5.20) hence says nothing about this particular entry, and instead becomes a statement only about the  $(N^2 - 1) \times (N^2 - 1)$  submatrices. In fact, since it has already been demonstrated that the rep of the algebra given by the  $(N^2 - 1) \times (N^2 - 1)$  submatrices of the adjoint is an irrep, the result obtained in Section 5.4 follows, and all of the submetrics  $\eta_{ab}^{(\rho)}$  will be proportional to

some canonical Killing metric  $\eta_{ab}$ . Since this includes the invertible metric  $\eta_{ab}^{(fn)}$ , it follows that the canonical Killing metric is itself invertible as well.

This fact implies that the rep-independent quadratic casimir  $Q$  can be constructed, and therefore, together with  $T_0$ , classifies irreps of  $U_q(\mathfrak{su}(N))$ . However, due to the block-diagonality of the Killing metric, it is immediately seen that the rep-independent quadratic casimir computed using only the primed generators, *i.e.*  $Q' := \eta^{ab} T'_a T'_b$ , may be written as

$$Q' = Q - \eta^{00} T_0^2. \quad (6.13)$$

Both terms on the right-hand side are central, so  $Q'$  is as well. Therefore, in classifying an irrep  $\rho$  of  $U_q(\mathfrak{su}(N))$ ,  $Q$  may be replaced by  $Q'$ .

### 6.3 Hopf Algebraic Structure

The primed generators are just linear combinations of the original ones, so they all have vanishing counit. Also, (5.28) indicates that  $f'_{Aa}{}^0 = 0$ , and therefore  $\mathfrak{U}^{\text{ad}} \mathfrak{g}' \subseteq \mathfrak{g}'$ .  $\mathfrak{g}'$  hence is a subspace of  $\mathfrak{U}$  which satisfies criteria (II) and (IV) of the definition of a QLA. Furthermore, as was shown in 6.1, it is spanned by traceless generators and its  $(N^2 - 1)$ -dimensional adjoint rep is an irrep. These facts lead to the results in Section 6.2, in which it was found that the Killing metric and quadratic casimir computed using only the elements of  $\mathfrak{g}'$  have all the properties which one would expect for the analogous quantities for the undeformed Lie algebra  $\mathfrak{su}(N)$ , *i.e.* invertibility, centrality, *etc.* Also,  $T_0 \rightarrow 0$  in the classical limit  $U_q(\mathfrak{su}(N)) \rightarrow U(\mathfrak{su}(N))$ .

All of this certainly seems to suggest that it is possible to follow the classical example, and define the QLA  $U_q(\mathfrak{su}(N))$  to be that subspace of  $\mathfrak{U}$  generated by  $\mathfrak{g}'$ . Unfortunately, this isn't the case; if the reader recalls (5.26), s/he sees that the primed generators do *not* close under multiplication— $T_0$  makes an appearance. It can also be shown that  $T_0$  appears explicitly in the coproducts of the primed generators as well, so  $U_q(\mathfrak{g}')$  doesn't satisfy (I) or (III).  $T_0$  must be included as a generator, and therefore one cannot follow the classical example and obtain  $\mathfrak{su}(N)$  from  $\mathfrak{gl}(N)$  simply by throwing away the “traceful” central element after imposing the hermiticity condition. (In contrast, at the QG level, there is a way to obtain  $SU_q(N)$  and preserve the HA structure [14].)

However, from the point of view of an irrep  $\rho$  of  $U_q(\mathfrak{su}(N))$ , the exact HA structure is not so important; the coproduct, counit and antipode will

not enter into the discussion, and  $T_0$  may be replaced by  $\mu(\rho)$  in relations like (5.26). The other good quantum numbers will be the ones obtained from considering  $\mathfrak{g}'$ , namely  $Q'$  and the weights associated with the Cartan subalgebra of  $\mathfrak{g}'$ , as in the classical case. Of course, if one wants to find higher reps from the tensor product of two irreps, the HA structure is absolutely vital, and will generally give results different from the classical case (*e.g.* if  $N = 2$ , and  $\rho_j$  is the spin  $j$  irrep,  $\rho_{j_1} \otimes \rho_{j_2}$  may no longer be  $\rho_{|j_1-j_2|} \oplus \dots \oplus \rho_{j_1+j_2}$ , but some deformed version instead).

## 6.4 Other Quantum Lie Algebras

Recall a very important assumption made in the discussion of  $U_q(\mathfrak{su}(N))$ , namely, that the numerical R-matrix satisfies a quadratic characteristic equation. This lead to the result that all of the traces  $\tau_A(\rho)$  were proportional to one another, and thus the primed generators (except  $T_0$ , of course) were traceless in all reps. It also lead to an explicit form of  $\mathcal{D}^A$ , which then implied the irreducibility of the rep  $ad'$ . Once these were in hand, the canonical Killing metric and the like followed.

For the QG  $SU_q(N)$ , this was in fact the case; however, what about the other QLAs corresponding to the deformed versions of the classical compact Lie algebras  $\mathfrak{so}(N)$  and  $\mathfrak{sp}(\frac{N}{2})$ ? As given in [6], the numerical R-matrices of  $SO_q(N)$  and  $SP_q(\frac{N}{2})$  satisfy the *cubic* characteristic equations

$$(\hat{R} - qI)(\hat{R} + q^{-1}I)(\hat{R} - \epsilon q^{\epsilon-N}) = 0, \quad (6.14)$$

where  $\epsilon = 1$  for  $SO_q(N)$  and  $\epsilon = -1$  for  $SP_q(\frac{N}{2})$ . Therefore, the condition that  $V_2$  and  $\hat{R}^2$  commute does *not* imply that  $V_2$  commutes with  $\hat{R}$ .

However, recall that the existence of a quadratic characteristic equation was not essential; what was necessary was the existence of  $\widetilde{M}$ , which for an arbitrary QLA means that one must be able to find the matrix  $(\widetilde{RPR})$ . At the present time, the author does not know if this matrix always exists for QLAs other than  $U_q(\mathfrak{su}(N))$ ; however, by using (6.4) (which still gives the fundamental rep if the appropriate numerical R-matrix is plugged in),  $\tau_A(fn)$  may be calculated for these other QLAs, with the results being

$$\tau_{(ij)}(fn) = (q^{\epsilon-N} - q^{N-\epsilon}) \delta_j^i. \quad (6.15)$$

So, even though it has not been shown for *all* reps, this at least seems to hint that the same results may be obtained for  $U_q(\mathfrak{so}(N))$  and  $U_q(\mathfrak{sp}(\frac{N}{2}))$  as for  $U_q(\mathfrak{su}(N))$ , and therefore the same conclusions follow.

This is not the whole story, however; for  $U_q(\mathfrak{su}(N))$ , an irreducible  $(N^2 - 1)$ -dimensional adjoint rep was easily interpreted as a deformation of the classical case. However, the adjoint reps for  $\mathfrak{so}(N)$  and  $\mathfrak{sp}(\frac{N}{2})$  have dimensions given by  $\frac{1}{2}N(N - \epsilon)$ , due to the fact that both are endowed with a metric  $C$  and an accompanying orthogonality restriction on their elements. This metric also exists in the deformed case, and obviously must be taken into account when discussing the adjoint rep. The QLA of these groups has been studied (*cf.* [16]), but how to extend the results of this work to such QLAs remains an open problem.

## 7 Example: $U_q(\mathfrak{su}(2))$

In this Section, some explicit calculations for the case of the QLA  $U_q(\mathfrak{su}(2))$  will be presented, and will (hopefully) serve to illustrate many of the results from the previous sections.

Consider the unital associative algebra generated by the elements  $\{H, X_+, X_-\}$ , modulo the Jimbo-Drinfel'd commutation relations [17, 18]

$$\begin{aligned} [H, X_{\pm}] &= \pm 2X_{\pm}, \\ [X_+, X_-] &= \frac{q^H - q^{-H}}{\lambda}, \end{aligned} \quad (7.1)$$

where  $q \in \mathbb{R}$ . This algebra is actually a QHA: the coproducts, counits, antipodes, and conjugates of these elements are given by

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, & \Delta(X_{\pm}) &= X_{\pm} \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes X_{\pm}, \\ \epsilon(H) &= \epsilon(X_{\pm}) = 0, \\ S(H) &= -H, & S(X_{\pm}) &= -q^{\pm 1}X_{\pm}, \\ \theta(H) &= H, & \theta(X_{\pm}) &= X_{\mp}, \end{aligned} \quad (7.2)$$

and the universal R-matrix by [19]

$$\mathcal{R} = \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{\frac{1}{2}(H \otimes H + nH \otimes 1 - n1 \otimes H)} X_+^n \otimes X_-^n, \quad (7.3)$$

where the “quantum factorial” is defined as

$$[n]_q! := \begin{cases} 1 & n = 0, \\ \prod_{m=1}^n [m]_q & n = 1, 2, \dots \end{cases} \quad (7.4)$$

This QHA is usually denoted by  $U_q(\mathfrak{su}(2))$ , and is the deformed version of the classical  $\mathfrak{su}(2)$  Lie algebra.

The fundamental reps for both the deformed and undeformed cases coincide, *i.e.* the matrices

$$fn(H) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad fn(X_+) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad fn(X_-) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad (7.5)$$

satisfy the Jimbo-Drinfel'd commutation relations for any value of  $q$ . For this rep, one can obtain the numerical R-matrix

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad (7.6)$$

and the matrices  $L^\pm$  from (2.17):

$$L^+ = \begin{pmatrix} q^{-\frac{1}{2}H} & -q^{-\frac{1}{2}}\lambda X_+ \\ 0 & q^{\frac{1}{2}H} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{\frac{1}{2}H} & 0 \\ q^{\frac{1}{2}}\lambda X_- & q^{-\frac{1}{2}H} \end{pmatrix}. \quad (7.7)$$

One may now find the QLA from this QHA by taking the above expressions for  $L^\pm$  and constructing the  $2 \times 2$  matrix  $X$ . The generators  $T_1$ ,  $T_+$ ,  $T_-$  and  $T_2$  are defined to be the entries of  $X$ , via

$$X = \begin{pmatrix} T_1 & T_+ \\ T_- & T_2 \end{pmatrix}. \quad (7.8)$$

(Notice from the hermiticity condition that  $T_1$  and  $T_2$  are self-adjoint, and  $\theta(T_\pm) = T_\mp$ .)

Using the change of basis introduced earlier, it is easily seen that  $T_0 = T_1 + \frac{1}{q^2}T_2$  and  $T'_\pm = T_\pm$ . However, the connection with the classical case is a bit more obvious if, instead of  $T'_1$ , one defines  $T_3 := q^2 T'_1 = \frac{1}{[2]_{q^{-1}}}(T_1 - T_2)$ . Using this basis, the adjoint actions are

$$T_0 \text{ad} T_0 = 0, \quad T_a \text{ad} T_0 = 0, \quad T_0 \text{ad} T_a = -\lambda [2]_{q^{-1}} T_a \quad (7.9)$$



(where  $a = +, -, 3$ ), as well as

$$\begin{aligned} T_3^{\text{ad}} \triangleright T_3 &= -\lambda T_3, & T_{\pm}^{\text{ad}} \triangleright T_{\mp} &= \pm \frac{[2]_{q^{-1}}}{q} T_3, \\ T_3^{\text{ad}} \triangleright T_{\pm} &= \pm q^{\mp 1} T_{\pm}, & T_{\pm}^{\text{ad}} \triangleright T_3 &= \mp q^{\pm 1} T_{\pm}. \end{aligned} \quad (7.10)$$

The commutation relations may also be found, albeit with a bit more work, since  $\widehat{\mathbb{R}}$  is a  $16 \times 16$  matrix. However, when all is said and done, one finds that  $T_0$  is indeed central as expected, and the other generators satisfy

$$\begin{aligned} q^{\mp 1} T_3 T_{\pm} - q^{\pm 1} T_{\pm} T_3 &= \pm \left( 1 - \frac{\lambda}{[2]_{q^{-1}}} T_0 \right) T_{\pm}, \\ T_+ T_- - T_- T_+ &= \frac{[2]_{q^{-1}}}{q} \left( 1 - \frac{\lambda}{[2]_{q^{-1}}} T_0 \right) T_3 + \frac{\lambda [2]_{q^{-1}}}{q} T_3^2. \end{aligned} \quad (7.11)$$

(Note that if  $T_0$  is replaced by  $-\lambda \left[ \frac{1}{2} \right]_q \left[ \frac{3}{2} \right]_{q^{-1}} I$ , and then the redefinitions

$$q \rightarrow q^{-\frac{1}{2}}, \quad T_{\pm} \rightarrow \sqrt{q^{\frac{1}{2}} \frac{[2]_q}{[2]_{q^{\frac{1}{2}}}}} T_{\pm}, \quad T_3 \rightarrow \frac{1}{\sqrt{[2]_q}} T_0, \quad (7.12)$$

are made, then one recovers the commutation relations for  $SU_q(2)$  in [20]. As the reader will see in the next subsection,  $T_0$  does indeed have this value in the fundamental irrep.)

In the following two subsections, the two obvious reps to consider, namely, the fundamental and adjoint, will be examined in detail.

## 7.1 Fundamental Representation

The numerical matrices for the generators  $\{T_1, T_+, T_-, T_2\}$  in the  $2 \times 2$  fundamental rep  $fn$  of  $U_q(\mathfrak{su}(2))$  may be found by using (6.4), where  $R$  is given by (7.6). Then, by switching the basis as described above, one obtains the matrices

$$\begin{aligned} fn(T_0) &= -\lambda \left[ \frac{1}{2} \right]_q \left[ \frac{3}{2} \right]_{q^{-1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & fn(T_3) &= \frac{1}{[2]_{q^{-1}}} \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{q^2} \end{pmatrix}, \\ fn(T_+) &= \begin{pmatrix} 0 & 0 \\ -\frac{1}{q} & 0 \end{pmatrix}, & fn(T_-) &= \begin{pmatrix} 0 & -\frac{1}{q} \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (7.13)$$

(Note that as previously argued,  $T_0$  does in fact vanish as  $q \rightarrow 1$ .) It is of course necessary to find the matrix  $D$ , *i.e.*  $u$  in this rep. By using the explicit form of  $R$  and the material in the Appendix, this is a simple calculation, with the result being

$$fn(u) = q^{-\frac{5}{2}} \begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}. \quad (7.14)$$

The  $4 \times 4$  Killing metric using all the generators will be block-diagonal, with the (00) entry being  $\eta_{00}^{(fn)} = q^{-\frac{1}{2}} [2]_{q^{-1}} \left[ \frac{1}{2} \right]_q^2 \left[ \frac{3}{2} \right]_{q^{-1}}^2$  and the  $3 \times 3$  metric computed in the basis  $\{T_+, T_-, T_3\}$  given by

$$\eta_{ab}^{(fn)} = q^{-\frac{7}{2}} \begin{pmatrix} 0 & q & 0 \\ \frac{1}{q} & 0 & 0 \\ 0 & 0 & \frac{q}{[2]_{q^{-1}}} \end{pmatrix}. \quad (7.15)$$

This must be proportional to the canonical Killing metric, of course, but one is free to choose the normalization (and thus define the index). The choice in this work will be

$$\eta_{ab} := \left( q + \frac{1}{q} \right) \begin{pmatrix} 0 & q & 0 \\ \frac{1}{q} & 0 & 0 \\ 0 & 0 & \frac{q}{[2]_{q^{-1}}} \end{pmatrix}, \quad (7.16)$$

so that  $\mathcal{I}(fn) = \frac{q^{-\frac{9}{2}}}{[2]_{q^{-1}}}$ . The rep-independent quadratic casimir thus has the form

$$Q' = \frac{1}{q^2 [2]_{q^{-1}}} \left( q^2 T_+ T_- + T_- T_+ + [2]_{q^{-1}} T_3^2 \right). \quad (7.17)$$

(If the substitutions (7.12) are once again made, then this agrees with the  $SU_q(2)$  casimir in [20] up to an overall factor of  $\frac{q^2+1}{q(q+1)^2}$ .) When the matrix reps for the generators are put into this expression, one finds

$$fn(Q') = \frac{[3]_{q^{-1}}}{[2]_q [2]_{q^{-1}}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (7.18)$$

Recall that the classical expressions for the index and quadratic casimir in the spin  $j$  irrep are  $\frac{1}{3}j(j+1)(2j+1)$  and  $j(j+1)I$  respectively, and so for the fundamental rep, *i.e.*  $j = \frac{1}{2}$ ,  $\mathcal{I}(fn) = \frac{1}{2}$  and  $fn(Q') = \frac{3}{4}I$ . For  $q = 1$ , the reader can immediately see that the results just obtained agree precisely with these values.

## 7.2 Adjoint Representation

Using the structure constants for  $U_q(\mathfrak{su}(2))$  from (7.10), it is trivial to find the generators in the  $3 \times 3$  adjoint rep  $ad'$ ; in the basis  $\{T_+, T_-, T_3\}$ , they take the forms

$$\begin{aligned} ad'(T_0) &= -\lambda [2]_{q^{-1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & ad'(T_3) &= \begin{pmatrix} \frac{1}{q} & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \\ ad'(T_+) &= \begin{pmatrix} 0 & 0 & -q [2]_{q^{-1}} \\ 0 & 0 & 0 \\ 0 & \frac{1}{q} & 0 \end{pmatrix}, & ad'(T_-) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q} [2]_{q^{-1}} \\ -\frac{1}{q} & 0 & 0 \end{pmatrix} \end{aligned} \quad (7.19)$$

and, again using the methods from the Appendix,  $u$  is

$$ad'(u) = \begin{pmatrix} \frac{1}{q^2} & 0 & 0 \\ 0 & \frac{1}{q^6} & 0 \\ 0 & 0 & \frac{1}{q^4} \end{pmatrix}. \quad (7.20)$$

$\eta_{00}^{(ad')} = \frac{1}{q^2} \lambda^2 [2]_{q^{-1}}^2 [3]_{q^{-1}}$  in this case, and the  $3 \times 3$  Killing metric is

$$\eta_{ab}^{(ad')} = \frac{[4]_{q^{-1}}}{q^3} \begin{pmatrix} 0 & q & 0 \\ \frac{1}{q} & 0 & 0 \\ 0 & 0 & \frac{q}{[2]_{q^{-1}}} \end{pmatrix} \quad (7.21)$$

so that  $\mathcal{I}(ad') = \frac{[4]_{q^{-1}}}{q^4 [2]_{q^{-1}}}$ . The rep-independent quadratic casimir in the adjoint rep is

$$ad'(Q') = \frac{[4]_{q^{-1}}}{[2]_{q^{-1}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7.22)$$

Once again, for  $q = 1$ , both  $\mathcal{I}(ad')$  and  $ad'(Q')$  agree with the classical values in the adjoint rep ( $j = 1$ ).

(The author suspects that the general forms for the two central quantities  $T_0$  and  $Q'$  in the spin  $j$  irrep  $\rho_j$  are

$$\rho_j(T_0) = -\lambda [j]_q [j+1]_{q^{-1}} I, \quad \rho_j(Q') = \frac{[2j]_q [2(j+1)]_{q^{-1}}}{[2]_q [2]_{q^{-1}}} I, \quad (7.23)$$

but unfortunately he has not been able to come up with a general proof for either. Oh well.)

A final comment: if one instead chooses the basis  $\{T_-, \sqrt{\frac{[2]_q-1}{q}}T_3, T_+\}$  for the matrix form of the canonical Killing metric, then

$$\eta_{ab} \propto \begin{pmatrix} 0 & 0 & \frac{1}{q} \\ 0 & 1 & 0 \\ q & 0 & 0 \end{pmatrix}. \quad (7.24)$$

Interestingly, this is the metric for the QG  $SO_{q^2}(3)$  [6]. There is some evidence that the classical equivalence between the groups  $SO(3)$  and  $SU(2)$  extends to a “deformed equivalence” between the QGs  $SO_{q^2}(3)$  and  $SU_q(2)$  [21], and the result here supports this.

## 8 Conclusions

In considering a QHA  $\mathfrak{U}$ , the presence of the universal R-matrix  $\mathcal{R}$  implies the existence of the element  $u$ . It was then shown in Section 5 that, together with a rep of  $\mathfrak{U}$ , this permits the introduction of a Killing form with properties (symmetry, invariance under the adjoint action) corresponding to those used to define such an object for the familiar classical case. If the QHA is also a QLA generated by  $\mathfrak{g}$ , the Killing metric is just given by plugging two basis elements of  $\mathfrak{g}$  into the Killing form.

Under certain assumptions such as invertibility and irreducibility of the adjoint rep, this Killing metric allows for the definitions of quantities which are present in the undeformed case, like quadratic casimirs, the canonical Killing metric, and the index of a rep. These assumptions are valid for the classical compact Lie algebras, and it was demonstrated that they also hold for the particular case of the QLA  $U_q(\mathfrak{su}(N))$ , but that the classical results are reproduced only for the  $(N^2 - 1)$ -dimensional subspace  $\mathfrak{g}'$  spanned by the traceless generators.

Therefore, if one wants to construct a physical theory in which the matter lives in irreps of  $U_q(\mathfrak{su}(N))$ , the states would be labeled by the quantum numbers corresponding to the values of  $T_0$  and  $Q'$  in that rep, as well as

by the weights from the deformed Cartan subalgebra. For example, in the particular case of  $U_q(\mathfrak{su}(2))$ , the doublet living in the fundamental irrep (“up” and “down” quarks, maybe?) would consist of the states (with the ordering  $|T_0, Q', T_3\rangle$ )

$$|\pm\rangle := \left| -\lambda \left[ \frac{1}{2} \right]_q \left[ \frac{3}{2} \right]_{q^{-1}}, \frac{[3]_{q^{-1}}}{[2]_q [2]_{q^{-1}}}, \pm \frac{q^{\mp 1}}{q [2]_{q^{-1}}} \right\rangle, \quad (8.1)$$

and the triplet living in the adjoint irrep would be

$$|\pm\rangle := \left| -\lambda [2]_{q^{-1}}, \frac{[4]_{q^{-1}}}{[2]_{q^{-1}}}, \pm q^{\mp 1} \right\rangle, \quad |0\rangle := \left| -\lambda [2]_{q^{-1}}, \frac{[4]_{q^{-1}}}{[2]_{q^{-1}}}, -\lambda \right\rangle. \quad (8.2)$$

As  $q \rightarrow 1$ , the latter two quantum numbers approach their classical values, and the quantum number given by  $T_0$  vanishes, as it must, since for the usual case of  $\mathfrak{su}(2)$ ,  $T_3$  and  $Q'$  form a maximal set of commuting operators. Interestingly, however, there may exist circumstances where the value of  $T_0$  actually removes a degeneracy; this would occur if, for some value of  $q$ , there are two states with the same quantum numbers given by the quadratic casimirs and weights, but different ones given by  $T_0$ . Then even though these two states would coincide in the  $q = 1$  case, they are distinct for the deformed case, and would be distinguishable particles in a theory with a deformed symmetry group.

In the context of Yang-Mills theories, the formalism here developed will also come in handy; if  $F \equiv T_A F^A$  is the QLA-valued field strength 2-form, and  $x = \xi^A T_A \in \mathfrak{g}$  (where the coordinates  $\xi^A$  are 0-forms), then the fact that  $\triangleright^{\text{ad}}$  is an action implies that

$$\begin{aligned} \text{tr}_\rho \left( u \left( x \triangleright^{\text{ad}} (F \wedge \star F) \right) \right) &= \text{tr}_\rho \left( u(x_{(1)}) \triangleright^{\text{ad}} F \wedge \star(x_{(2)}) \triangleright^{\text{ad}} F \right) \\ &= \xi^A \epsilon(T_A) \text{tr}_\rho (u F \wedge \star F) \\ &= 0, \end{aligned} \quad (8.3)$$

so if  $F \mapsto x \triangleright^{\text{ad}} F$  describes an infinitesimal gauge transformation, as it does in the undeformed case, the above trace is gauge-invariant. Thus, the quantity  $\eta_{AB}^{(\rho)} F^A \wedge \star F^B$  may be used in the action as the kinetic energy of the gauge field.

For the case of  $U_q(\mathfrak{su}(N))$  (and perhaps others as well), where the Killing metric is block-diagonal in the primed basis, then by writing  $F = F^0 T_0 + F^a T'_a$ , the action takes the form

$$S = \int \left( \eta_{ab}^{(\rho)} F^a \wedge \star F^b + \eta_{00}^{(\rho)} F^0 \wedge \star F^0 \right) + \dots, \quad (8.4)$$

which, for the particular example where  $N = 2$  and  $\rho = ad$ , is

$$\begin{aligned} S = & \frac{[4]_{q^{-1}}}{q^3} \int \left( q F^+ \wedge \star F^- + \frac{1}{q} F^- \wedge \star F^+ + \frac{q}{[2]_{q^{-1}}} F^3 \wedge \star F^3 \right) \\ & + \frac{\lambda^2}{q^2} [2]_{q^{-1}}^2 [3]_{q^{-1}} \int F^0 \wedge \star F^0 + \dots \end{aligned} \quad (8.5)$$

Because  $T_0$ , and thus  $\eta_{00}^{(\rho)}$ , vanishes in the classical limit, the term involving  $F^0$  will not be present in the undeformed case of  $\mathfrak{su}(N)$ , but must be included for  $q \neq 1$  in order to guarantee invariance under the adjoint action. In fact, since the subspace  $\mathfrak{g}'$  is the part of  $\mathfrak{U}$  which survives in the classical limit, the term quadratic in  $F^0$  might be better interpreted as an interaction term in the lagrangean rather than part of the kinetic energy. This view is further supported by the fact that since  $\eta_{00}^{(\rho)}$  will be small when  $q$  is close to 1, it may be thought of as a perturbation parameter.

These examples demonstrate that from the point of view of physics, the existence of a deformed Killing metric (at least for  $U_q(\mathfrak{su}(N))$ ) provides additional structure which may in fact have consequences when constructing a theory with a quantum symmetry. In fact, it is possible that there may be ways to retain some of this extra structure even in the classical case (*e.g.* if  $\lambda F^0$  is kept nonzero in the  $U_q(\mathfrak{su}(2))$  Yang-Mills example above). This, however, remains an open question, and (in the author's opinion) worth further study.

## Acknowledgements

I would like to thank Chryss Chryssomalakos, Oleg Ogievetsky, Peter Schupp and Bruno Zumino for helpful comments and suggestions. I would also like to mention Olaf Backofen, who brought the connection with [20] to my attention.

## A Appendix: The Matrix $D$

If  $\rho$  is a rep of a QHA  $\mathfrak{U}$ , and  $\mathfrak{A}$  is the associated QG, the numerical matrix  $D$  is defined to be  $\rho(u)$ , where  $u$  is the element defined by (2.11):

$$D^i_j := \alpha \langle u, A^i_j \rangle \quad (\text{A.1})$$

( $\alpha$  is just a overall normalization constant). Several results follow immediately: first of all, an explicit computation using the definition of  $u$  leads to the result

$$I = \alpha \text{tr}_1(D_1^{-1} \hat{R}^{-1}) = \alpha^{-1} \text{tr}_2(D_2 \hat{R}), \quad (\text{A.2})$$

where  $\text{tr}_J$  is shorthand for the trace over the  $J^{th}$  pair of indices, *e.g.* the  $(^i_j)^{th}$  element of the rightmost expression in the above equation is  $\alpha^{-1} D^m_n \hat{R}^{in}_{jm}$ . These relations can be “inverted” in the sense of solving them for  $D$  and  $D^{-1}$  by using the following: given an  $N^2 \times N^2$  numerical matrix  $M$ , one may define the matrix  $\tilde{M} = [(M^{t_1})^{-1}]^{t_1}$  ( $t_J$  denotes transposing with respect to the  $J^{th}$  pair of indices) which satisfies

$$M^{im}_{n\ell} \tilde{M}^{nk}_{jm} = M^{mi}_{\ell n} \tilde{M}^{kn}_{mj} = \delta^i_j \delta^k_\ell. \quad (\text{A.3})$$

Hence, by using (A.2), one can find expressions for  $D$  and its inverse:

$$D = \alpha \text{tr}_2(P \tilde{R}), \quad D^{-1} = \alpha^{-1} \text{tr}_2(P(\tilde{R}^{-1})). \quad (\text{A.4})$$

Recall that the element  $c := uS(u)$  is central in  $\mathfrak{U}$ ; therefore, in an irrep,  $\rho(c)$  must be proportional to the unit matrix. Hence, define the constant  $\beta$  by means of the identity

$$\langle c, A^i_j \rangle = (\alpha\beta)^{-1} \delta^i_j. \quad (\text{A.5})$$

Using the explicit expressions for  $c$  and  $u$ , one obtains

$$I = \beta^{-1} \text{tr}_1(D_1^{-1} \hat{R}) = \beta \text{tr}_2(D_2 \hat{R}^{-1}), \quad (\text{A.6})$$

or, by “inverting”,

$$D = \beta^{-1} \text{tr}_1(P(\tilde{R}^{-1})), \quad D^{-1} = \beta \text{tr}_1(P \tilde{R}). \quad (\text{A.7})$$

The dual version (in the QG  $\mathfrak{A}$ , that is) of (2.12) is

$$S^2(A) = DAD^{-1}. \quad (\text{A.8})$$

This identity, the definition of the  $D$ -matrix, and (2.14) give

$$(D^{-1})^t A^t D^t S(A)^t = S(A)^t (D^{-1})^t A^t D^t = 1, \quad (\text{A.9})$$

and (2.14) and (A.9) together then imply that

$$\tilde{R} = D_1^{-1} R^{-1} D_1 = D_2 R^{-1} D_2^{-1}. \quad (\text{A.10})$$

Then from either this relation or the fact that  $(S^2 \otimes S^2)(\mathcal{R}) = \mathcal{R}$ , it follows that

$$D_1 D_2 R = R D_1 D_2. \quad (\text{A.11})$$

The properties of  $D$  just described imply the following: if  $M$  is an  $N \times N$  matrix, then

$$\begin{aligned} \text{tr}_1(D_1^{-1} R^{-1} M_1 R)^{i_j} &= \text{tr}_1(D_1^{-1} R_{21} M_1 R_{21}^{-1})^{i_j} \\ &= \text{tr}(D^{-1} M) \delta_j^i. \end{aligned} \quad (\text{A.12})$$

Also, if the elements of  $M$  commute with the elements of  $A$ ,

$$\text{tr}(D^{-1} S(A) M A) = \text{tr}(D^{-1} M). \quad (\text{A.13})$$

In particular, if  $M$  is a matrix on which  $\mathfrak{A}$  right coacts via  $\Delta_{\mathfrak{A}}(M^i_j) = M^k_\ell \otimes S(A^i_k) A^\ell_j$ , then (A.9) implies

$$\Delta_{\mathfrak{A}}(\text{tr}(D^{-1} M)) = \text{tr}(D^{-1} M) \otimes 1. \quad (\text{A.14})$$

For this reason,  $\text{tr}(D^{-1} M)$  is called the *invariant trace* of  $M$ .

For QLAs, this matrix makes its appearance in the following way: the matrix  $\tilde{\mathbb{R}}$ , defined by means of (A.3) using the numerical R-matrix  $\hat{\mathbb{R}}$  of the QLA, is given by

$$S(O_C^A)^{\text{ad}} T_D = \tilde{\mathbb{R}}^{AB}{}_{CD} T_B. \quad (\text{A.15})$$

Notice that (4.2) implies that

$$\begin{aligned} S^2(T_A) &= S^2(O_A^B) T_C S(O_B^C) \\ &= S(O_A^B)^{\text{ad}} T_B \\ &= \mathbb{D}^B{}_A T_B, \end{aligned} \quad (\text{A.16})$$



where the numerical matrix  $\mathbb{D}$  is defined through (A.15) as

$$\mathbb{D}^A{}_B = \tilde{\mathbb{R}}^{CA}{}_{BC}, \quad (\text{A.17})$$

which is precisely the same as (A.4) in QLA language. Similarly,  $\mathbb{D}$  satisfies

$$\begin{aligned} S^2(O_A{}^B) &= \mathbb{D}^C{}_A O_C{}^D (\mathbb{D}^{-1})^B{}_D, \\ \mathbb{D}_1 \mathbb{D}_2 \hat{\mathbb{R}} &= \hat{\mathbb{R}} \mathbb{D}_1 \mathbb{D}_2, \\ \tilde{\mathbb{R}}^{AB}{}_{CD} &= (\mathbb{D}_1^{-1} \hat{\mathbb{R}}^{-1} \mathbb{D}_2)^{AB}{}_{DC}. \end{aligned} \quad (\text{A.18})$$

## References

- [1] M. E. Sweedler, *Hopf Algebras*, Benjamin Press, 1969
- [2] E. Abe, *Hopf Algebras*, Cambridge University Press, 1977
- [3] S. Majid, *Int. J. Mod. Phys. A* **5** 1 (1990)
- [4] S. L. Woronowicz, *Commun. Math. Phys.* **111** 613 (1987)
- [5] V. G. Drinfel'd, in: *Proceedings, Intl. Congress Math.* 798, Berkeley, 1986
- [6] N. Yu. Reshetikhin, L. A. Takhtadzhyan and L. D. Faddeev, *Leningrad Math. J.* **1** 193 (1990)
- [7] P. Schupp, P. Watts and B. Zumino, *Commun. Math. Phys.* **157** 305 (1993)
- [8] S. L. Woronowicz, *Commun. Math. Phys.* **122** 125 (1989)
- [9] D. Bernard, *Prog. Theor. Phys. Suppl.* **102** 49 (1990)
- [10] B. Jurčo, *Lett. Math. Phys.* **22** 177 (1991)
- [11] N. Yu. Reshetikhin and M. A. Semenov-Tian-Shansky, *Lett. Math. Phys.* **19** 133 (1990)
- [12] B. Zumino, in: *Proceedings, Mathematical Physics X* 20, Springer-Verlag, 1992
- [13] N. Jacobson, *Lie Algebras*, Dover Publications, 1979
- [14] P. Schupp, P. Watts and B. Zumino, *Lett. Math. Phys.* **25** 139 (1992)
- [15] P. Schupp, P. Watts and B. Zumino, *Adv. Appl. Cliff. Alg. (Proc. Suppl.)* **4** (S1) 125 (1994)
- [16] U. Carow-Watamura, M. Schlieker, S. Watamura and W. Weich, *Commun. Math. Phys.* **142** 605 (1991)
- [17] M. Jimbo, *Int. J. Mod. Phys. A* **4** 3759 (1989)

- [18] V. G. Drinfel'd, *Sov. Math. Dokl.* **32** 254 (1985)
- [19] M. Rosso, *Commun. Math. Phys.* **124** 307 (1989)
- [20] E. Witten, *Nucl. Phys. B* **330** 285 (1990)
- [21] B. Zumino, private communication